

De Sitter Space and Holography

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Abstract

Our Universe is expanding. Driven by a tiny positive cosmological constant, this expansion is predicted to accelerate until the Cosmos becomes an ultra-cold de Sitter Universe. There is observational evidence that a similar era of exponential expansion, called inflation, happened once before in the extremely dense Universe directly after the Big Bang. A theory of quantum gravity is necessary to explain how inflation gave rise to everything we currently observe in the sky, and to give a microscopic description of the fundamental temperature and entropy of our future Universe. Decades of work in quantum gravity and string theory has revolutionized our understanding of these problems. In particular the realization of the holographic principle in Anti-de Sitter universes (AdS), which provides tractable and conjecturally UV-complete theories of quantum gravity in terms of lower-dimensional conformal field theories (CFTs). Nevertheless, we currently do not understand any UV-complete models of an expanding universe beyond perturbation theory. In this thesis, we analyze holographic theories of AdS and dS and focus on the construction of a complete model with dynamical gravity and a four-dimensional de Sitter vacuum.

The first part is dedicated to the proposed AdS/CFT correspondence. We first review the (quantum) gravity side and the field theory side of this duality separately. After describing the general conjecture, we proceed to construct new black hole configurations in a particular model, in the regime where the AdS side of the duality has a valid description as a supergravity. This model is a consistent truncation of the bosonic sector of a four-dimensional $\mathcal{N} = 2$ supersymmetric compactification of M-theory. The black hole solutions we find, have electric and magnetic charges and are surrounded by massive vector and scalar fields. We study the black hole thermodynamics and phase transitions in the canonical ensemble and discuss the implications for the dual CFT.

In part two we review elements of the physics of de Sitter space, first classically and then supplemented with canonically quantized scalar field theory. We discuss the wave function description of the Universe and its proposed

expression as a no-boundary path integral. These concepts are used to review the dS/CFT proposal and compare it to the AdS duality. We analyze the wave function in terms of CFT quantities and formulate conditions under which it describes classical time evolution. We find that they are more constraining than previous conditions derived in quantum cosmology. Afterwards, specific realizations of the dS/CFT conjecture are reviewed, which relate Vasiliev theory of higher-spin gravity to Euclidean CFTs with anti-commuting scalars. The spectrum of primary operators is outlined and some holographic calculations of the partition function are reviewed, highlighting a problem with the wave function interpretation thereof. We then analyze whether the most general correlation function of scalar field theory in a fixed de Sitter background can be obtained from the CFT in a suitable limit. We find that they are not, in particular because the CFT appears to contain no mechanism to generate non-vanishing correlation functions of commutators.

In the third part of this thesis, we propose a model of the Hilbert space of de Sitter quantum gravity in terms of fermionic operators. This model contains manifestly non-vanishing (anti)-commutation relations and a positive definite inner product, designed to address the previous problem of a probabilistic interpretation. The gauge symmetry group of this model is the same as that of the aforementioned Euclidean CFT and the construction of gauge invariant operators is analogous. We find that it is possible to characterize the gauge invariant Hilbert space states and operators in terms of (bosonic) functions on a compact manifold, which has the symplectic structure of a phase space. A related quantum mechanical model of Grassmann matrices, supplied with non-trivial dynamics, is analyzed. In the low energy sector, the model is accurately described in terms of the associated classical phase space. Finally, we study whether the field operators of perturbative quantized Vasiliev theory can be identified within the Hilbert space. We argue that the correct bosonic commutation relations can be obtained, in a particular sector of the Hilbert space, by a Holstein-Primakoff transformation. The implications of this proposal are analyzed both analytically and numerically. We discuss the breakdown of the perturbative regime and conclude by outlining some of the open questions and challenges faced by this proposal and de Sitter holography in general.

Beknpte samenvatting

Ons Universum dijt uit. Deze expansie wordt verwacht te versnellen wegens de minuscule kleine kosmologische constante die de Kosmos uiteindelijk zal doen evolueren naar een ultra-koud *de Sitter* Universum. Er zijn observationele aanwijzingen dat een gelijkaardig tijdperk van exponentiële expansie, *inflatie* genoemd, al eens eerder plaatsvond in het extreem geconcentreerde Universum net na de Big Bang. We hebben een theorie van kwantumzwaartekracht nodig om te verklaren hoe inflatie aan de oorsprong kan liggen van alles wat we momenteel in het heelal observeren, alsook voor een microscopische beschrijving van de fundamentele temperatuur en entropie van ons toekomstig Universum. Decennia vol ontwikkelingen in mogelijke kwantumzwaartekrachttheorieën, zoals snaartheorie, hebben ons revolutionair nieuwe inzichten gegeven in dit vraagstuk. In het bijzonder beschikken we nu over een realisatie van het holografisch principe in *Anti-de Sitter* ruimte (AdS) die begrijpbare en vermoedelijk UV-complete kwantumzwaartekrachttheorieën beschrijft aan de hand van lagerdimensionale conforme-veldentheorieën (CFT's). Voor expanderende universa daarentegen, is er momenteel geen enkel UV-compleet model dat we (behalve via storingsrekening) goed begrijpen. In deze thesis analyseren we holografische theorieën van AdS en dS en focussen we op de constructie van een UV-compleet model met zwaartekracht en met een vierdimensionaal de Sitter vacuüm.

Het eerste deel is gewijd aan het AdS/CFT-voorstel. We bespreken enerzijds de kant met (kwantum)zwaartekracht en anderzijds de veldentheoriekant van deze dualiteit. De algemene hypothese wordt dan uitgelegd en toegepast op de constructie van nieuwe soorten zwarte gaten in een specifiek model in het regime waar de AdS-kant van de dualiteit een accurate beschrijving heeft als een superzwaartekracht. Dit model is een consistente truncatie van de bosonische sector van een vierdimensionale $\mathcal{N} = 2$ supersymmetrische compactificatie van M-theorie. De zwarte gaten die we vinden hebben zowel elektrische en magnetische ladingen en zijn omringd door een massief vectorveld en scalaire velden. We bestuderen de zwart-gat-thermodynamica en faseovergangen in het

canonisch ensemble en bespreken de implicaties voor de duale CFT.

In deel twee behandelen we enkele elementen van de fysica van de Sitter-ruimte, eerst klassiek en daarna met een canonisch gekwantiseerd scalair veld. We bespreken de golf functiebeschrijving van het Universum en de voorgestelde uitdrukking daarvan als een “no-boundary” padintegraal. We geven een overzicht van het dS/CFT-voorstel, waarop de voorgaande concepten worden toegepast, en vergelijken het met de AdS dualiteit. We analyseren de golf functie in termen van CFT-grootheden en formuleren voorwaarden waaronder ze klassieke tijdsevolutie beschrijft. Deze blijken strenger te zijn dan voorwaarden die voordien waren afgeleid in kwantumkosmologie. Naderhand bespreken we specifieke realisaties van de dS/CFT hypothese die Vasiliev’s theorie van hogere-spin zwaartekracht relateren aan Euclidische CFT’s met anti-commuterende scalairen. We leiden het spectrum van primaire operatoren af en schetsen enkele holografische berekeningen van de partitiefunctie, met klemtoon op de problematische golf functieinterpretatie ervan. Daarna analyseren we of de meest algemeen mogelijke correlatiefunctie van scalaire veldentheorie in een onbeweeglijk de Sitter universum kan worden bekomen vanuit de CFT in een passend regime. We stellen vast dat dit niet mogelijk is, in het bijzonder omdat de CFT geen mechanisme bevat om een niet-triviale verwachtingswaarde van commutatoren te genereren.

In het derde deel van deze thesis stellen we een model voor om de Hilbertruimte van de Sitter kwantumzwaartekracht te beschrijven aan de hand van fermionische operatoren. Dit model bevat niet-triviale anti-commutatierelaties en een positief definit inproduct, ontworpen om de problematische probabilistische interpretatie op te lossen. De ijsymmetriegroep van dit model is dezelfde als die van de bovenvermelde Euclidische CFT en de constructie van ijsymmetrische operatoren is analoog. Het blijkt mogelijk om de Hilbertruimte te karakteriseren aan de hand van bosonische functies op een compacte variëteit die de symplectische structuur heeft van een faseruimte. Een gerelateerd kwantummechanisch model van Grassmann-matrices en niet-triviale dynamica wordt eveneens geanalyseerd. In de lage-energiesector wordt het model accuraat beschreven door een bosonische theorie op de geassocieerde klassieke faseruimte. Tenslotte bestuderen we of de veldoperatoren van perturbatief gekwantiseerde Vasiliev theorie geïdentificeerd kunnen worden binnen de Hilbertruimte. We beargumenteren dat, in een bepaalde sector van de Hilbertruimte, de correcte bosonische commutatierelaties kunnen worden bekomen via een Holstein-Primakofftransformatie. De implicaties van dit voorstel worden zowel numeriek als analytisch onderzocht. We bespreken de beperkingen van het perturbatieve regime en besluiten met een overzicht van enkele open vragen en uitdagingen voor dit voorstel en voor de Sitter-holografie in het algemeen.

List of Abbreviations

Λ-CDM	The Λ -Cold Dark Matter Model (Λ denotes the cosmological constant), also known as the Standard Model of Cosmology. 2
AAdS	Asymptotically anti-de Sitter space. 14
ABJM	$\mathcal{N} = 6$ superconformal Chern-Simons theory described by Aharony, Bergman, Jafferis and Maldacena [9]. 25
ADM	Arnowitt-Deser-Wheeler formalism. 68 , 86
AdS	Anti-de Sitter space. 3 , 7 , 25 , 53 , 105 , 115 , 181
AHS	Anninos-Hartman-Strominger model [21]. 106 , 123 , 127 , 182
BF	Breitenlohner-Freedman bound. 14
BH	Black hole. 8
BPS	Bogomol'nyi-Prasad-Sommerfield bound (also used to refer to states satisfying this bound). 25 , 181
CFT	Conformal field theory. 3 , 7 , 25 , 53 , 77 , 105 , 115 , 127 , 168 , 181
dS	De Sitter space. 2 , 53 , 77 , 115 , 127 , 181
FG	Fefferman-Graham expansion. 72 , 87
GR	General Relativity. 2 , 7 , 68

HP	Holstein-Primakoff. 171
IR	Infra-red, referring to low-energy phenomena. 36, 111
KG	Klein-Gordon inner product. 60
NBWF	No-boundary wave function of the Universe. 65, 79
NP	North pole. 56
ODE	Ordinary differential equation. 27
QFT	Quantum field theory. 2, 16, 94
RG	Renormalization group. 16, 73, 111
RR	Ramond-Ramond field. 26
SM	Standard Model of Particle Physics. 2
SP	South pole. 56, 82
TP	Turning point. 83
UV	Ultra-violet, referring to high-energy phenomena. 2, 28, 53, 94, 128, 167
WDW	Wheeler-DeWitt equation. 78
WKB	Wentzel-Kramers-Brillouin approximation. 78, 175

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Chapter 1

Introduction

The aim of physics is to analyze the complicated phenomena in the world around us, in order to discover emerging patterns. We are fortunate to find that, more than mere patterns, there are fundamental *laws of physics*. Combined with the right initial conditions, they completely determine the evolution of everything we observe. It is therefore useful to study physics, to discover what these rules are and how to use them to describe and predict our observations. Countless experimental and theoretical breakthroughs have allowed the field of physics to progress spectacularly. Scientists are now able to probe the Universe with increasingly precise measurements on scales ranging from sub-nuclear to cosmological. There is an array of subfields of physics which cover these scales, each with its own experimental focus and theoretical regime of validity.

This thesis aims to contribute to the subfields of *high-energy physics* and *cosmology*. Both of these are a direct result of two important breakthroughs in the early twentieth century: the development of quantum mechanics and relativity. They have altered our conception of the basic notions of space and time, and of matter and forces.

1.1 Two Standard Models and a spacetime

The revolution triggered by these two breakthroughs has led to two Standard Models of our Universe that agree remarkably well with a vast amount of experimental data.

The theory of quantum mechanics was used in particle physics and

generalized to the framework of quantum field theory (QFT). This ultimately lead to the Standard Model of Particle Physics (SM) describing electromagnetism and the nuclear forces, as well as all known elementary matter particles, in a unified framework. A lot of important breakthroughs were realized, in particular the formalism of regularization and renormalization which are essential for the theory to make sensible predictions. These predictions have been checked extensively, mainly in particle accelerator experiments, and confirmed to be very accurate.

The principle of relativity, on the other hand, has lead to the development of the theory of General Relativity (GR), which describes gravity by introducing the notion of spacetime as a dynamic variable. This lead to the realization that our Universe is not eternal and to the development of cosmology, the study of how the cosmos itself evolves in time. The Standard Model of Cosmology, called the Λ -CDM model, currently explains the cosmological evolution of our Universe in agreement with a wide array of astrophysical observations. Our Universe is observed to be expanding. This is described in the Λ -CDM by a positive *cosmological constant* of the order of $1.5 \cdot 10^{-25} \text{ kg/m}^3$. It predicts that the future of our Universe will look ever more like the *de Sitter* (dS) solution to GR. The same solution is also used to describe how an inflationary era in the earliest, very dense stage of the Universe set the stage for the cosmos we observe today. The primary goal of this thesis is to understand the physics of de Sitter space.

The two standard models are famously incompatible. GR is a classical field theory on which the procedure of canonical quantization does not yield a renormalizable field theory. It therefore only provides an effective field theory that consistently describes low energy phenomena. The theory is not reliable when high energy, ultra-violet (UV) effects are involved. The measured value of the cosmological constant seems unnaturally tiny within the theory of gravity at low energies. The theory indicates that de Sitter space has a very low intrinsic temperature, as well as a very large entropy. However, it does not provide an explanation in terms of the microstates of a more fundamental theory. Furthermore, it is not capable of consistently explaining the process of inflation, which is sensitive to UV effects.

To answer these questions about our Universe and to unify the two Standard Models, we need to find and understand a *UV-complete* theory of quantum gravity in de Sitter space.

1.2 Quantum gravity

The accurate predictions of the two Standard Models are both a triumph and a concern. The lack of contradicting experimental observations makes it hard to find new clues for what a unified theory of physics should look like. Nevertheless, there is important progress that can be made within theoretical high-energy physics itself: the basic requirement of internal theoretical consistency currently provides some of the most stringent constraints on a putative unifying theory.

The development of string theory is a good illustration of the constraining power of internal consistency. Decades of theoretical work changed its scope from a model of the confining strong nuclear force to a proposed UV-complete theory of quantum gravity. It was realized, by analyzing the internal consistency of the theory, that the original model is part of a larger theory. Several sectors of the full theory are separately well-defined, well-studied, and related to each other by a “web of dualities” that currently makes up our understanding of string theory.

An important result of the sequence of revolutionary insights in string theory is the conjecture known as the AdS/CFT duality. The proposal asserts that certain conformal field theories (CFTs) provide a UV-complete description of gravitational theories in asymptotically Anti-de Sitter spacetimes¹ (AdS). Within string theory, the analysis in [4] manages to isolate a sector that can be described either as a gravitational theory in AdS or as a CFT on a fixed spacetime. This sector decouples from other parts of the theory, therefore forming a UV-complete theory of quantum gravity by itself. After the discovery of this first model, the AdS/CFT correspondence was analyzed more generally and extended to a framework with a *dictionary* that translates quantities on both sides of the duality [5, 6, 7, 8]. In particular, the CFT correlation functions can be calculated on the supergravity side as the variations of the on-shell action with respect to boundary conditions of the fields at spatial infinity. Numerous other explicit models were discovered within (ten-dimensional) string theory, as well as some models outside of string theory [9, 10].

One common property of these models is their *holographic* nature: the spacetime on which the CFT is defined has less dimensions than the dynamical spacetime on the gravity side of the duality. This is an implementation of a characteristic property of quantum gravity called the *holographic principle*, which had been discovered before AdS/CFT based the analysis of the thermodynamic properties of black holes [11, 12, 13, 14, 15].

Even though these holographic theories were found in an attempt to describe

¹ We will describe this class of spacetimes in more detail in Chapter 2.

the microscopic (quantum) properties of interactions between elementary particles, they depend crucially on the global structure of spacetime. This links the problem of finding a UV-complete theory of quantum gravity to cosmology. As mentioned before, our observable Universe is *not* asymptotically AdS [16, 17]. The possibility of a holographic duality for asymptotically de Sitter spacetimes was investigated [18, 19, 20]. It was observed that some essential properties of the AdS/CFT proposal, such as matching symmetry groups on both sides of the duality, have a counterpart in dS. The first entries in the dictionary of the dS/CFT framework were written: different types of CFT operators were matched to different types of bulk fields, and bulk correlation functions were shown to match the properties of CFT correlation functions. Important differences were noted as well. For example, the holographically emergent direction in dS/CFT is not a spatial one but time. Even though the dS/CFT framework was set up, no string theory constructions similar to [4] could be found to construct an explicit model.

Nevertheless, an explicit model that realizes this dS/CFT conjecture was recently proposed [21]. The gravitational side of the proposal is a four-dimensional asymptotically de Sitter spacetime containing an infinite number of massless fields. The CFT contains a large number of fermionic scalar fields, violating the spin-statistics theorem. The spectrum of CFT operators matches that of fields in the bulk.

1.3 Motivation and strategy for this thesis

The primary goal of this thesis is to contribute to our understanding of the physics in de Sitter space. We want to work towards a *complete quantum theory with dynamical gravity and a four-dimensional de Sitter vacuum*.

While a large part of this thesis contributes towards this primary goal, it is not our only motivation. We will study several aspects of holography and obtain results which are important in their own right. Accordingly, the structure of this thesis is not a single argument that leads to the completion of our primary goal. Chapters 3, 5 and 9 consist of papers published independently. The organization of this thesis is summarized by three parts:

- I Analysis and application of the AdS/CFT correspondence (Chapters 2 - 3),
- II Analysis of the dS/CFT correspondence, its elements and explicit realization (Chapters 4 - 7),

III Proposal and development of a complete theory of quantum gravity in de Sitter (Chapters 8 - 10).

In Part I, Chapter 2, we review the AdS/CFT conjecture. It is a more well-established correspondence than any current proposal for dS holography. To study the latter, it is important to know which elements from AdS carry over to dS, and which are different. This is by far not the only motivation to study AdS/CFT and undersells its importance in its own right. The AdS duality has been used to understand aspects of particle physics and condensed matter physics in regimes of strong coupling, where perturbative methods fail [22, 23, 24, 25]. This is the context of Chapter 3, where we analyze a model that has been used to describe disordered, glassy systems through AdS/CFT [26]. We find new types of black hole solutions in AdS supergravity, which are surrounded by non-trivial matter field profiles.

Part II of this thesis is initiated in Chapter 4, where we outline the elements and the statement of dS/CFT. We compare and contrast the situation with that in AdS. We observe an essential difference, namely that the dS/CFT proposal does not specify the Hilbert space of the bulk theory. Chapter 5 is dedicated to an important notion used in the formulation of the dS duality, the no-boundary wave function of the Universe. We work out how it can be formulated in terms of the natural quantities of dS/CFT, namely the coefficients that parameterize the spacetime and matter fields in the asymptotic future. In terms of these coefficients, we obtain a criterion for the wave function to predict classical evolution in de Sitter space. While the dS/CFT correspondence provides a motivation for this calculation, the result is independent of de Sitter holography. In Chapter 6, we review the first explicit model of dS/CFT [21], which allows to do explicit calculations to check the dS/CFT proposal. The interpretation of the CFT partition function as a wave function is challenged by some explicit calculations [27]. As a final chapter in Part II, we analyze how bulk locality is encoded in the CFT. We identify certain linear combinations of CFT primaries which represent local field operators in dS. However, we will find that the CFT correlation functions do not reproduce to the most general bulk field correlation functions.

Based on these observations, we propose to search for a complete theory in which the Hilbert space of the dS theory is explicit. This is the content of Part III. The proposed Hilbert space contains fermionic scalar fields transforming in the same gauge group as the model of [27]. Chapter 8 starts with the definition of the Hilbert space we consider. We associate a compact classical phase space to the model and characterize states and operators in terms of functions on this manifold, for which there is a well-defined inner product. In Chapter 9, the formalism of Chapter 8 is applied to the quantum mechanics of

Grassmann matrices. We supply the system with a Hamiltonian and analyze its dynamics. The chapter is concluded with a comment on potential holographic applications of this model towards gravitational theories with finite entropy. Chapter 10 is dedicated identifying the operators in the Hilbert space proposal which correspond to the field operators of canonically quantized Vasiliev theory in the regime where the perturbative description is accurate. This identification is only possible in a restricted sense, which we interpret as the breakdown of perturbation theory within this complete proposed model of quantum gravity.

The analysis of the model defined in Chapter 8 leaves open many important questions. This is the subject of the concluding Chapter 11.

Chapter 2

Anti-de Sitter holography

This chapter is dedicated to the [AdS/CFT](#) correspondence. We start in §2.1 by reviewing the development of black hole thermodynamics, a discovery that lead to an essential concept underlying AdS/CFT: the *holographic principle*. In §2.2, we introduce the Anti-de Sitter spacetime and discuss its symmetries, coordinate systems and other special properties which will be relevant to the proposal of AdS/CFT. The formalism of conformal field theory is reviewed in §2.3. Because of its importance in later chapters, we will introduce general properties before we discuss aspects that are specific to their application in AdS/CFT. In the last section, §2.4, we review the AdS/CFT proposal.

2.1 Black holes and thermodynamics

Black holes are solutions of General Relativity with an event horizon. They are regions in spacetime from which no timelike trajectory leads away. Physically, they are be formed when a spatial region bounded by an area of size $A = 4\pi R^2$ contains more mass than $2G_N/R$, where G_N is Newton’s gravitational constant. Depending on the position, velocity, mass and angular momentum of the collapsing matter configuration, there is a unique black hole solution in [GR](#) that describes the resulting black hole. If Maxwell’s theory of electromagnetism is coupled to GR, the black hole can also have an electric charge and is described by the Kerr-Newman solution¹ [28, 29, 30]. A number of additional “no-hair”

¹ In this introduction, we are assuming that the cosmological constant vanishes and spacetime approaches Minkowski space far away from the black hole. The possible black hole solutions are different for other values of the cosmological constant, and the no-hair theorems

theorems were proven for GR coupled to different forces and matter fields [31, 32, 33, 34, 35].

These uniqueness theorems and the fact that event horizon act as a “perfect unidirectional membrane” [36] shielding everything inside the black hole, raise the question how an observer outside the black hole is to interpret the laws of thermodynamics and statistical mechanics. It was argued in [11, 37] that black holes obey these laws with an entropy proportional to the area A and temperature given by the surface gravity of the black hole (BH). A semiclassical calculation in quantum field theory in the background of a collapsing black hole fixed the constants of proportionality [12, 13]

$$S_{\text{BH}} = \frac{k_{\text{B}} A}{4l_{\text{P}}^2} , \quad (2.1)$$

where k_{B} is the Boltzmann constant and l_{P} is the Planck length, defined in terms of the speed of light c , Newton’s gravitational constant G and the reduced Planck constant \hbar as $l_{\text{P}} = \sqrt{\hbar G/c^3} \approx 1.6 \cdot 10^{-35} \text{m}$ in SI units². The *Bekenstein-Hawking entropy* S_{BH} was furthermore argued to be the maximal entropy – as a notion of microscopic degrees of freedom – that can be assigned to a region of space in a gravitational theory [38, 39].

The fact that this upper limit for the entropy is proportional to the area around the system, and not its volume, has been interpreted as an indication that the number of fundamental degrees of freedom in quantum gravity are related to the area of surfaces in spacetime, rather than volumes [15, 14]. It is called the holographic principle and is believed to be a fundamental property of quantum gravity. It realized in the AdS/CFT correspondence, as we will outline below.

2.2 Anti-de Sitter space (AdS)

Anti-de Sitter spacetime is a solution to Einstein’s equations with a negative cosmological constant Λ . AdS_{d+1} exists for any positive number d of spatial dimensions and one time direction. It is a maximally symmetric spacetime, which means that (in any coordinates) its Riemann curvature tensor is given in

do not necessarily apply. We will consider a novel example of a black hole “with hair” in Chapter 3.

² From here on we will use units in which the speed of light $c = 1$ and Planck’s constant $\hbar = 1$.

terms of the metric tensor as

$$R_{\mu\nu\rho\sigma} = -\frac{1}{l_{\text{AdS}}^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) , \quad \Lambda = -\frac{d(d-1)}{2l_{\text{AdS}}^2} , \quad (2.2)$$

where l_{AdS} has units of length and is called the “AdS radius”. Another property of maximally symmetric spacetimes (indeed a property that alternatively be used as their definition) is the presence of $d(d+1)/2$ isometries.

2.2.1 Global AdS – periodic

There are number of common coordinate representations of AdS_{d+1} . The one in which the symmetries are most manifest and which explains Figure 2.1 is its *embedding* in $d+2$ -dimensional flat space with coordinates X^M and metric

$$ds_{d+2}^2 = \tilde{\eta}_{MN} dX^M dX^N , \quad \tilde{\eta}_{MN} = \text{diag}(-1, 1, \dots, 1, -1) . \quad (2.3)$$

The indices used in this thesis are explained in Table 2.1. The $d+1$ -dimensional one-sheeted hyperboloid satisfying the following constraint in this embedding space defines the geometry of AdS,

$$\tilde{\eta}_{MN} X^M X^N = -l_{\text{AdS}}^2 . \quad (2.4)$$

The symmetries of the embedding space that are preserved by the hypersurface are rotations around the origin, generated by³

$$L_{MN} = X_M \partial_N - X_N \partial_M , \quad [L_{MN}, L_{PQ}] = 4\tilde{\eta}_{[N[P} L_{M]Q]} . \quad (2.5)$$

These are the generators of the $so(2, d)$ algebra.

Index	Range	Example
M, N	$0, 1, \dots, d, d+1$	X^M in (2.3)
I, J	$1, \dots, d, d+1$	X^I in (4.3)
μ, ν	$0, 1, \dots, d-1, d$	x^μ in (2.2)
α, β	$0, 1, \dots, d-1$	x^α in (2.10)
i, j	$1, \dots, d-1, d$	x^i in (2.7)
a, b	$1, \dots, d-1$	θ^a in (2.6)

Table 2.1: Conventions for indices and coordinates.

³ We use square brackets to denote the anti-symmetric combination of indices, with a factor of $1/n!$ where n is the number of indices. In this case, we have $4\tilde{\eta}_{[N[P} L_{M]Q]} = \tilde{\eta}_{NP} L_{MQ} - \tilde{\eta}_{NM} L_{PQ} - \tilde{\eta}_{QP} L_{MN} + \tilde{\eta}_{QM} L_{PN}$. Round brackets around indices will denote symmetrization with the same convention.

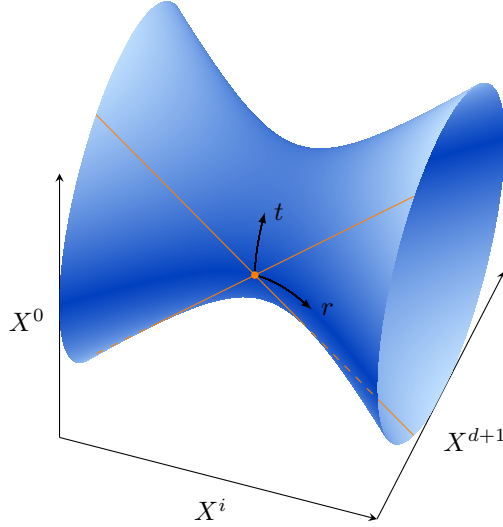


Figure 2.1: Three-dimensional slice of AdS_{d+1} embedded as a hyperboloid in three-dimensional flat space (2.4). According to the metric (2.3), the X^0 and X^{d+1} directions contribute negatively to the line element and all other directions contribute positively. The latter are all equivalent and indicated collectively as X^i . The global coordinates t and r are indicated, as well as the light cone originating from the origin and propagating within AdS.

It is possible to describe this space using only $d + 1$ coordinates (t, r, θ^a) called *global coordinates*. The angles θ_a , with $0 \leq \theta^1 < 2\pi$ and all other θ between 0 and π , are a set of coordinates on the hypersphere S^{d-1} . By itself, this hypersphere can be embedded in d -dimensional flat space. Consider Cartesian coordinates $\omega^i \in \mathbb{R}^d$, then the sphere of unit radius corresponds to the hypersurface $\delta_{ij}\omega^i\omega^j = 1$ and is parameterized by the angles θ^a as

$$\begin{aligned}
 \omega^1 &= \sin \theta^1 \sin \theta^2 \dots \sin \theta^{d-1} \\
 \omega^2 &= \cos \theta^1 \sin \theta^2 \dots \sin \theta^{d-1} \\
 \omega^3 &= \cos \theta^2 \dots \sin \theta^{d-1} \\
 &\dots \\
 \omega^{d-1} &= \cos \theta^{d-1} .
 \end{aligned} \tag{2.6}$$

The flat space metric induces the line element $d\Omega_{d-1}^2$ on the hypersphere, which can be written using induction: $d\Omega_1^2 = (d\theta^1)^2$ and $d\Omega_d^2 = (d\theta^d)^2 + \sin^2(\theta^d)d\Omega_{d-1}^2$.

Using these coordinates, we can write the relation between global coordinates on AdS_{d+1} and the embedding space coordinates,

$$X^0 = \sqrt{l^2 + r^2} \sin \frac{t}{l}, \quad X^i = r \omega^i, \quad X^{d+1} = \sqrt{l^2 + r^2} \cos \frac{t}{l}, \quad (2.7)$$

as well as AdS metric induced by (2.3) onto the hyperboloid,

$$ds^2 = - \left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2. \quad (2.8)$$

By the nature of the embedding, the light-cone of any point on the AdS hypersurface – i.e. the collection of all spacetime points null-separated from that point – is the intersection of the embedding space light-cone with the hyperboloid. It can be seen from the definition of the time coordinate t or directly from Figure 2.1 that any light ray will reach *spatial infinity in finite time* t .

This observation can be made more readily by introducing a radial coordinate ρ for which $\cos \rho = 1/\sqrt{1 + r^2/l^2}$. It takes values from 0 at the origin to $\pi/2$ at spatial infinity $r \rightarrow \infty$. The metric (2.8) is now given by

$$ds^2 = \frac{1}{\cos^2 \rho} [-dt^2 + l^2(d\rho^2 + \sin^2 \rho d\Omega_{d-1}^2)] . \quad (2.9)$$

Since light rays are insensitive to the radius-dependent prefactor, they will cover the distance from the origin to spatial infinity in a time-span $\Delta t = l\pi/2$, as depicted in Figure 2.2. This is a first sign of the special nature of AdS which will come in useful when formulating quantum gravity: observers at any two spatial points in AdS can communicate through light rays in finite time.

The locus $\rho = \pi/2$ is not strictly part of AdS. However, it is possible to define the *conformal compactification of AdS* which does include these points at spatial infinity [40]. The boundary of this conformal completion is usually called the *conformal boundary of AdS*. Since massless excitations can travel from this boundary to anywhere in the bulk in finite time, the Cauchy problem composed of classical equations of motion and initial conditions on an equal time slice cannot be extended beyond that time unless they are supplemented by boundary conditions on the conformal boundary of AdS.

In summary, AdS is like a test-tube: all points in AdS can interact within finite time and it is possible control the behavior in the bulk by choosing the boundary conditions.

2.2.2 Global AdS – unfolded

As it was introduced before, AdS is perfectly periodic in t . This is necessary to embed it in a $d + 2$ dimensional flat space and draw a picture like Figure 2.1. Periodicity in t is not, however, a necessary condition to satisfy Einstein’s equations with a negative cosmological constant. Since they are local differential equations, they are not sensitive to global properties such as the topology of timelike curves. It is therefore equally valid to consider the *covering space* for which metric (2.8) or (2.9) has t ranging from $-\infty$ to ∞ without imposing periodicity. From here on, we will use the term “AdS” to refer to this eternal version of the spacetime. Topologically, this spacetime is a cylinder, as one can see from (2.9).

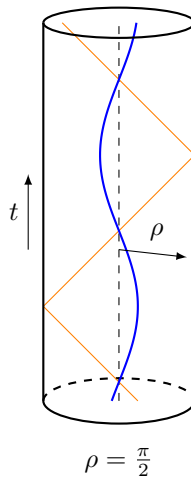


Figure 2.2: A conformal diagram of AdS with $l = 1$. This three-dimensional representation is obtained by taking the Penrose diagram and revolving it around the dashed vertical line. The trajectory of a light ray going through the origin and bouncing off the boundary at $\rho = \pi/2$ is shown in orange. The blue line describes the typical trajectory of a massive particle.

The embedded manifold picture of AdS remains useful still. For example, the geodesic motion of probe particles on the fixed AdS background – also described by a local differential equation – is periodic in t . This statement is independent of the mass of said particle. In particular in the limit of massless particles this is consistent⁴ with our earlier observation that light can reach spatial infinity in finite time.

⁴ At least insofar we can assume perfectly reflecting “boundary conditions” at $\rho = \pi/2$. Such boundary conditions may make sense but are not unique.

2.2.3 The Poincaré patch

Another useful set of coordinates are called *Poincaré coordinates* (z, x^α) . They do not cover the full space described by (2.8) – in fact only a geodesically incomplete part of it – but they have the advantage that the metric in these coordinates is conformally equivalent to flat space. More precisely,

$$ds^2 = \frac{l^2}{z^2} (dz^2 + \eta_{\alpha\beta} dx^\alpha dx^\beta) , \quad (2.10)$$

where $z > 0$ and $\eta_{\alpha\beta} = \text{diag}(-1, 1, \dots, 1)$. This system of coordinates can be obtained from the embedding space coordinates using the relations

$$\begin{aligned} X^0 &= \frac{lx^0}{z} , & X^d &= l \frac{1 - z^2 - \eta_{\alpha\beta} x^\alpha x^\beta}{2z} , \\ X^a &= \frac{lx^a}{z} , & X^{d+1} &= l \frac{1 + z^2 + \eta_{\alpha\beta} x^\alpha x^\beta}{2z} . \end{aligned} \quad (2.11)$$

It covers only the part of AdS for which $X^d + X^{d+1} > 0$ in the embedding space. In Figure 2.1, if the X^i direction is the X^d direction, this constraint corresponds to a diagonal plane that cuts through the hyperboloid.

In the limit $z \rightarrow 0$, this coordinate system covers part of the conformal boundary of AdS. The limit $z \rightarrow \infty$ is merely a coordinate singularity of the Poincaré coordinates. It traces out a null-hypersurface which is sometimes called the Poincaré horizon. Since it can be chosen freely and is not associated with any physical object in AdS, it is more similar to a Rindler horizon than to a Schwarzschild horizon.

2.2.4 Scalar field theory on the Poincaré patch

Before coupling gravity to matter and describing deviations from the pure AdS metric, it is useful to briefly consider the behavior of matter fields on a fixed AdS background. Specifically, we will be interested in the properties near the conformal boundary $z \approx 0$, which is not present in flat space. An extensive analysis in global AdS, with scalar and vector fields will follow in Chapter 3. For now, we will restrict to the example of a free scalar field ϕ of mass m^2 , propagating on the Poincaré patch with metric (2.10). The action and the corresponding equations of motion are given by

$$\begin{aligned} S &= -\frac{1}{2} \int \frac{dz d^d x}{z^{d+1}} [z^2 (\partial_z \phi)^2 + z^2 \eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + m^2 \phi^2] , \\ 0 &= z^{d+1} \partial_z (z^{1-d} \partial_z \phi) + z^2 \eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - m^2 \phi . \end{aligned} \quad (2.12)$$

The kinetic terms in all but the z direction are similar to those in Minkowski space. The eigenstates of the flat space Laplacian are Fourier modes. Let us therefore try to find solutions which are of the form $\phi(z, x) = \varphi_k(z)e^{ik_\alpha x^\alpha}$ (with k_0 independent from the k_a coefficients). The equations of motion turn into a Bessel-type equation for the z -dependent coefficients

$$z^{d+1}\partial_z(z^{1-d}\partial_z\varphi) - (z^2k^2 + m^2)\varphi = 0, \quad (2.13)$$

where $k^2 = \eta^{\alpha\beta}k_\alpha k_\beta$. The solutions to this equation are indeed Bessel functions,⁵ but it is possible to extract the behavior of the solution in the limit $z \rightarrow 0$ directly from this equation, by ignoring the term z^2k^2 which is suppressed in this limit [41]. To leading order, the solutions are powers z^Δ , where Δ solves $\Delta(\Delta - d) = m^2l^2$. More precisely,

$$\varphi(z) = z^{\Delta_-}[\alpha + \mathcal{O}(z^2)] + z^{\Delta_+}[\beta + \mathcal{O}(z^2)], \quad (2.14)$$

where $\Delta_\pm \equiv \frac{d}{2} \pm \nu$ and $\nu \equiv \sqrt{m^2l^2 + d^2/4}$.

For real values of Δ , the correspondence $\Delta(\Delta - d) = m^2l^2$ is bounded by $m^2l^2 \geq -d^2/4$. This includes a range of tachyonic scalar fields. In Minkowski space, fields with negative m^2 have perturbations which grow exponentially. In perturbative quantum field theory, this leads to an instability of the perturbative “vacuum”, which signals that it is not the true vacuum of the theory. However, it was argued in [42] that $m^2l^2 \geq -d^2/4$ is exactly the range of masses for which the energy of excitations is bounded from below, and no such pathologies occur. It is called the *Breitenlohner-Freedman* (BF) bound.

2.2.5 Asymptotically AdS spacetimes

The conformal boundary is a “robust” property of spacetimes with a negative cosmological constant. It takes an infinite energy perturbation to destroy the behavior of the metric (2.9) near $\rho \approx \pi/2$, since the volume of space is unbounded there. Furthermore, any massless excitation is redshifted an infinite amount when it reaches the conformal boundary. We can therefore consider a class of Asymptotically Anti-de Sitter (AAdS) spacetimes which have an asymptotic completion similar to AdS [43, 44]. In particular, they have the same conformal boundary as AdS.

The above was formulated in global coordinates, but an analogous definition is possible in the Poincaré patch. The statement is then that the metric is of

⁵ A similar equation arises for the scalar field in de Sitter space. We will do a more careful analysis of the exact solutions in Chapter 4.

the form

$$ds^2 = \frac{l^2}{z^2} [dz^2 + g_{\alpha\beta}(z, x) dx^\alpha dx^\beta] , \quad (2.15)$$

where $g_{\alpha\beta}$ is smooth and finite as $z \rightarrow 0$. It has an expansion in powers of z (and if d is even, there can also be logarithmic terms) with x^α -dependent coefficients $g_{(n)}(x)$ with n a nonnegative integer. This expansion is called the *Fefferman-Graham expansion* [43, 44]. The Einstein can be solved in the z -direction⁶ order by order in terms of the coefficients in terms of $g_{(0)}$ and $g_{(d)}$. This has been worked out in general in [45] and we will treat the analogous calculation in de Sitter space in Chapter 5.

When gravity is coupled to matter fields, the spacetime remains AAdS and the modes take the form (2.14) near the boundary as long as the coefficients of diverging modes vanish identically. For tachyonic scalar fields (above the BF bound), solutions are exactly of the form (2.14), but scalar fields with positive mass must have $\alpha(x) = 0$. We will consider a detailed example in Chapter 3, where we find new black hole solutions, surrounded by fields with a non-trivial radial profile, in AAdS supergravity.

2.3 Conformal field theory (CFT)

Many successful theories of physics make extensive use of symmetries to explain a broad range of phenomena in comprehensible terms. Symmetries lead to simplifications, perhaps most famously by the Noether formalism, which associates a conservation law to each continuous symmetry. Field theories provide an excellent illustration of how symmetries can guide the development of physics. One example is the standard model of particle physics, which successfully implements the formalism of quantum field theory with gauged *internal* symmetries. A second example is the Coleman-Mandula theorem [46] which tightly constrains *spacetime* symmetries. It states that the largest possible bosonic⁷ spacetime symmetry group of a non-trivial relativistic field theory is the conformal group.

⁶ This is not how field equations are usually solved – the z -direction does not correspond to time evolution – but it is perfectly possible as long as no caustics are encountered.

⁷ The Coleman-Mandula theorem has an important generalization called the Haag-Lopuszanski-Sohnius theorem [47], which allows for symmetries relating fields of different statistics, fermions and bosons. The conformal group can then be extended to the *superconformal group*, which still allows for non-trivial relativistic field theories. The large symmetry group can be used to restrict the possible theories of supersymmetry and supergravity: another impressive example of the power of symmetries.

In this section, we will define the conformal group and consider field theories which have it as their symmetry group. We will find that this determines a lot of their properties. CFTs find their application in holography – as we will describe below – but also at the critical points of statistical mechanics models [48] and as the fixed points of the renormalization group (RG) flow of quantum field theories [49, 50]. Interestingly, it is possible that a non-field theoretic model of statistical mechanics has the same CFT description as the fixed point of some QFT. This remarkable universality at critical points is a strong motivation to study CFTs. Notably the conformal bootstrap program aims to map out the set of possible CFTs and has made considerable progress, both analytically and numerically [51, 52, 53, 54].

2.3.1 The conformal group

The *conformal group* on the Riemannian plane⁸ is the set of invertible maps $x \rightarrow x'$ which map $\delta_{ij} dx^i dx^j$ onto a multiple of itself [55, 48]. Such mappings are called conformal because they leave angles invariant: for any two tangent vectors v^i and w^i at a specific point of the spacetime manifold, the quantity $v \cdot w / (|v||w|)$ is invariant. The following transformations are in this group:

- translations $x'^i = x^i + a^i$ along a constant vector a ,
- rotations around the origin $x'^i = \Lambda^i_j x^j$ with $\Lambda \in SO(d)$,
- dilatations, or scale transformations $x'^i = \lambda x^i$,
- special conformal transformations

$$x'^i = \frac{x^i - b^i x^2}{1 - 2b_j x^j + b^2 x^2} . \quad (2.16)$$

In more than two dimensions, all other conformal maps are combinations of the transformations in this list [55, 48].

Infinitesimal transformations $x'^i - x^i \equiv \delta x^i = -\epsilon^i$ can be represented as vector fields $\epsilon(x) = \epsilon^i(x) \partial_i$ on the manifold. The metric transforms as $\delta g_{ij} = 2\partial_{(i} \epsilon_{j)}$. This is a multiple of itself whenever ϵ equals any of

$$\begin{aligned} P_i &= \partial_i , & M_{ij} &= x_j \partial_i - x_i \partial_j , \\ D &= x^i \partial_i , & K_i &= 2x_i x^j \partial_j + x^2 \partial_i . \end{aligned} \quad (2.17)$$

⁸ The definition of the conformal group can be generalized to pseudo-Riemannian manifolds. For Minkowski space, this means that some of the rotations become boosts. The signature of the algebra in (2.19) changes to η_{MN} as in (2.3) and the conformal group is $SO(2, d)$.

The commutators of these generators satisfy the *conformal algebra* $so(1, d + 1)$, which can be most conveniently summarized by relabeling the generators as L_{MN} with indices $M, N = (0, i, d + 1)$ and

$$L_{ij} = M_{ij} , \quad L_{0,d+1} = D , \quad L_{0i} = \frac{P_i - K_i}{2} , \quad L_{d+1,i} = \frac{P_i + K_i}{2} . \quad (2.18)$$

In terms of these combinations, the commutation relations are indeed those of $so(1, d + 1)$,

$$[L_{MN}, L_{PQ}] = 4\eta_{[P[N}L_{M]Q]} , \quad \eta_{MN} = \text{diag}(-1, 1, \dots, 1) . \quad (2.19)$$

2.3.2 Field theories with conformal symmetry

The conformal group arises as the symmetry group of certain Euclidean quantum field theories. The correlation functions are invariant under a representation of the conformal group which acts on the field operators. This section follows the developments and most of the notation of [50].

Correlation functions can be given in different forms, for example by a path integral over fields $\phi(x)$ weighted by a (Euclidean) action $S[\phi]$

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \int \mathcal{D}\phi \, \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) e^{-S[\phi]} , \quad (2.20)$$

where the operators $\mathcal{O}_i(x_i)$ are given functions of the ϕ . The correlation functions could also be given as the time-ordered expectation values of quantum states in a Hilbert space, describing fields living on a foliation of \mathbb{R}^d , embedded with coordinates y^a and with a parameter σ labeling the leaves of the foliation,

$$\langle \mathcal{O}_1(\sigma_1, y_1) \dots \mathcal{O}_n(\sigma_n, y_n) \rangle = \langle 0 | \mathcal{T}[\hat{\mathcal{O}}_1(\sigma_1, y_1) \dots \hat{\mathcal{O}}_n(\sigma_n, y_n)] | 0 \rangle . \quad (2.21)$$

Here we use the bra-ket notation to denote the Hilbert space inner product and use \mathcal{T} to denote that the operators are to be arranged in order of decreasing σ .

If there is a quantum theory that gives rise to the conformally invariant correlation functions in this way, it is called a “quantization” of the theory [50]. A different foliation of \mathbb{R}^d would lead to a different quantization,⁹ which would nevertheless be equivalent because (by construction) it has the same correlation functions.

We can show that a QFT is invariant under conformal transformations if the stress tensor $T^{ij}(x)$ is traceless, symmetric and furthermore conserved in

⁹ See appendix A of [50] for an interesting, non-field theoretic example.

the following sense: for any set of local operators $\mathcal{O}_i(x)$,

$$\begin{aligned} & \frac{\partial}{\partial x^i} \langle T^{ij}(x) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle \\ &= - \sum_{m=1}^n \delta(x - x_m) \frac{\partial}{\partial x_m^j} \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle . \end{aligned} \quad (2.22)$$

As long as T^{ij} is not inserted at the same point as any of the other operators, (2.22) amounts to the usual current conservation equation $\partial_i T^{ij} = 0$. The presence of *contact terms*, i.e. the Dirac- δ function in (2.22), indicates that this correlation function identity should be interpreted as a distributional one and can be integrated over x . Consider for example an arbitrary surface Σ with codimension 1 and a vector field $\epsilon = \epsilon^i(x) \partial_i$. The quantity

$$Q_\epsilon(\Sigma) \equiv - \int_\Sigma dS_i \epsilon_j T^{ij} , \quad (2.23)$$

is *conserved* if it satisfies $\partial_i(\epsilon_j T^{ij}) = 0$ in the sense of (2.22). This means that the value of $Q_\epsilon(\Sigma)$ – when inserted in a particular correlation function – is the same as that for any other surface Σ' as long as Σ can be deformed continuously to Σ' without crossing any operators in the correlation function. For this reason, $Q_\epsilon(\Sigma)$ is called a *topological surface charge*. Using the properties of the stress tensor, the requirement of conservation can be rewritten as

$$0 = \partial_i(\epsilon_j T^{ij}) = \frac{1}{2} [\partial_i \epsilon_j + \partial_j \epsilon_i + \alpha(x) \delta_{ij}] T^{ij} . \quad (2.24)$$

This is guaranteed if the term in square brackets vanishes, which is the conformal Killing equation for ϵ . If the stress tensor was not traceless, α would have to vanish and this would reduce to the ordinary Killing equation. On flat space, it is satisfied whenever ϵ is any of the vector fields in (2.17). The conservation (2.22) of the symmetric and traceless stress tensor thus implies that the transformations generated by (2.17) are indeed symmetries of the correlation functions.

We will denote the action of a topological surface charge $Q_\epsilon(\Sigma)$ on a local operator $\mathcal{O}(x)$ with a hat. For example, an infinitesimal translation can be represented on a local operator as

$$\hat{P}_i \mathcal{O}(x) = \partial_i \mathcal{O}(x) , \quad e^{a^i \hat{P}_i} \mathcal{O}(x) = \mathcal{O}(x + a) . \quad (2.25)$$

We can find the consistent representations of the conformal group on fields by looking at the infinitesimal transformation of fields at the origin [48].

Under infinitesimal rotations around the origin, operators can transform in a representation of the $SO(d)$ subgroup as

$$\hat{M}_{ij} \mathcal{O}(0) = \mathcal{S}_{ij} \cdot \mathcal{O}(0) , \quad (2.26)$$

where for each value of i, j , \mathcal{S}_{ij} is the generator of a representation of $so(d)$ which is determined by the *spin* of the operator \mathcal{O} . For example, a scalar field transforms in the trivial representation $\mathcal{S}_{ij} = 1$. For vector fields \mathcal{O}^i , the matrix \mathcal{S}_{ij} is such that $\mathcal{S}_{jk}\mathcal{O}^i = (\delta_k^i\mathcal{O}_j - \delta_j^i\mathcal{O}_k)/2$. For a spinor field \mathcal{O}^u , where the index u ranges from 1 to $2^{\lfloor \frac{d}{2} \rfloor}$, the matrix \mathcal{S} correspond to second-rank γ -matrices of the d -dimensional Clifford algebra [56]. The action of rotations on operators away from the origin can be derived from the conformal algebra

$$\begin{aligned}\hat{M}_{ij}\mathcal{O}(x) &= e^{x^k\hat{P}_k}e^{-x^k\hat{P}_k}\hat{M}_{ij}e^{x^k\hat{P}_k}\mathcal{O}(0) \\ &= e^{x^k\hat{P}_k}[x_j\hat{P}_i - x_i\hat{P}_j + \hat{M}_{ij}]\mathcal{O}(0) \\ &= [x_j\partial_i - x_i\partial_j + \mathcal{S}_{ij}]\mathcal{O}(x) ,\end{aligned}\tag{2.27}$$

where we have used the Baker-Campbell-Haussedorff formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \frac{1}{6}[A, [A, [A, B]]] + \dots\tag{2.28}$$

To further characterize the operator, consider how $\mathcal{O}(0)$ can transform under scale transformations. In the conformally invariant field theories we will be interested in, there are operators which transform to multiples of themselves under dilatations,

$$\hat{D}\mathcal{O}(0) = \Delta\mathcal{O}(0) .\tag{2.29}$$

By an argument analogous to (2.27), it can be shown that the dilatation must act on operators away from the origin as

$$\hat{D}\mathcal{O}(x) = (x^i\partial_i + \Delta)\mathcal{O}(x) .\tag{2.30}$$

The representation of special conformal transformations on operators must be compatible with the conformal algebra. For example, if an operator $\mathcal{O}(0)$ has a well-defined conformal weight Δ , the operator $\hat{K}_i\mathcal{O}(0)$ must have weight $\Delta - 1$. Similarly, $\hat{P}_i\mathcal{O}(0)$ has weight $\Delta + 1$. In the theories we will discuss in later sections, there is a lower bound¹⁰ on the scaling dimensions. Therefore there are operators $\mathcal{O}(0)$ which are annihilated by special conformal transformations, $\hat{K}_i\mathcal{O}(0) = 0$, called *primary* operators. By acting with derivatives \hat{P}_i on primary operators, one creates the corresponding *descendants* which have ever higher conformal weights. Away from the origin, a primary operator transforms under infinitesimal special conformal transformations as

$$\hat{K}_i\mathcal{O}(x) = [2x_i(\Delta + x^j\partial_j) - x^2\partial_i - 2x^j\mathcal{S}_{ij}]\mathcal{O}(x) .\tag{2.31}$$

¹⁰ For example, we will encounter only CFTs for which all two-point functions fall to zero at large spatial distances. We will see in the next subsection that this requires $\Delta > 0$.

2.3.3 Correlation functions

The form of the correlation functions in a conformally invariant field theory is heavily constrained by conformal symmetry. For example, the two-point function of two scalar primary operators $\langle \mathcal{O}_1(x) \mathcal{O}_2(y) \rangle$ is not any arbitrary function of the $2d$ coordinates. To be invariant under translations and rotations, it must be a function $f(|x - y|)$ of only the distance between the two points. Furthermore, the invariance under dilatations requires

$$\begin{aligned} 0 &= \langle \hat{D} \mathcal{O}_1(x) \cdot \mathcal{O}_2(y) + \mathcal{O}_1(x) \cdot \hat{D} \mathcal{O}_2(y) \rangle \\ &= [(x^i \partial_{x^i} + \Delta_1) + (y^i \partial_{y^i} + \Delta_2)] f(|x - y|) \\ &= |x - y| f'(|x - y|) + (\Delta_1 + \Delta_2) f(|x - y|) . \end{aligned} \quad (2.32)$$

This is satisfied if $f = c|x - y|^{-\Delta_1 - \Delta_2}$ for any constant. A similar calculation for invariance under special conformal transformation determines that f vanishes whenever the conformal weights are unequal [55, 48, 49, 50]. The result is therefore

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(y) \rangle = \frac{c \delta_{\Delta_1, \Delta_2}}{|x - y|^{2\Delta_1}} . \quad (2.33)$$

Furthermore, three-point functions have the general form [57]

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{f_{123}}{x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} x_{23}^{\Delta_2 + \Delta_3 - \Delta_1} x_{31}^{\Delta_3 + \Delta_1 - \Delta_2}} , \quad (2.34)$$

where $x_{ij} \equiv |x_i - x_j|$.

The correlation functions of a CFT can be obtained from a *generating function* $Z[J]$, such that

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{\delta}{\delta J_1(x_1)} \dots \frac{\delta}{\delta J_n(x_n)} Z[J] \Big|_{J=0} . \quad (2.35)$$

In case the CFT correlation functions can be obtained from a path integral weighted by the exponential of an action S_0 , the generating function can be written as the path integral with the action $S = S_0 + \int d^d x J(x) \mathcal{O}(x)$. The J are called currents, dual to a respective primary operator \mathcal{O} .

2.3.4 Lorentzian field theories

In conventional Lorentzian quantum field theory, physical states of the system correspond to vectors in a Hilbert space which have unit norm, $\langle \psi | \psi \rangle = 1$. The

observables that can be measured in experiments are represented as Hermitian operators $\hat{\mathcal{O}}$ acting on the states,

$$\langle \psi | \hat{\mathcal{O}} | \psi \rangle \equiv \langle \psi | \hat{\mathcal{O}} \psi \rangle = \langle \hat{\mathcal{O}}^\dagger \psi | \psi \rangle = \langle \hat{\mathcal{O}} \psi | \psi \rangle , \quad (2.36)$$

where the second step is the definition of the Hermitian conjugate \dagger associated with this inner product. In particular, the expectation value of $\hat{\mathcal{A}}^\dagger \hat{\mathcal{A}}$ (where $\hat{\mathcal{A}}$ is any, non necessarily Hermitian operator) is positive since it is the norm of a state,

$$\langle \psi | \hat{\mathcal{A}}^\dagger \hat{\mathcal{A}} | \psi \rangle = \langle \hat{\mathcal{A}} \psi | \hat{\mathcal{A}} \psi \rangle \geq 0 . \quad (2.37)$$

In the Schrödinger picture, time translations are represented as unitary operators on the states. They are generated by a Hermitian Hamiltonian $H^\dagger = H$. In a Hilbert space, such a Hermitian operator has a complete set of eigenstates with real eigenvalues. If the system is in state $|\psi(0)\rangle$ at some time $t = 0$, the expectation value at time t of a local operator $\hat{\mathcal{O}}(x)$ is given by

$$\langle \psi(t) | \hat{\mathcal{O}}(x) | \psi(t) \rangle = \langle \psi(0) | e^{-iHt} \hat{\mathcal{O}}(x) e^{iHt} | \psi(0) \rangle . \quad (2.38)$$

In the Heisenberg picture, the time dependence is included not in the state but in the operator $\mathcal{O}(t, x) \equiv e^{-iHt} \hat{\mathcal{O}}(0, x) e^{iHt}$. Hermiticity of the original operator $\hat{\mathcal{O}}(0, x)$ implies that $\hat{\mathcal{O}}(t, x)^\dagger = \hat{\mathcal{O}}(t, x)$. If we Wick rotate to Euclidean time $\tau = it$, the operators are

$$\hat{\mathcal{O}}(\tau, x) = e^{-H\tau} \hat{\mathcal{O}}(0, x) e^{H\tau} , \quad \hat{\mathcal{O}}(\tau, x)^\dagger = \hat{\mathcal{O}}(-\tau, x) . \quad (2.39)$$

Euclidean operators which satisfy the second equation are called *real*. Combining this with (2.37), we see that a Euclidean field theory can only be the Wick rotation of a Lorentzian quantum field theory with a positive definite inner product, if it satisfies *reflection positivity*,

$$\langle \psi | \hat{\mathcal{O}}(-\tau, x) \hat{\mathcal{O}}(\tau, x) | \psi \rangle \geq 0 , \quad (2.40)$$

for all states ψ and operators real $\hat{\mathcal{O}}$.

A Lorentzian quantum field theory living on a cylinder (a spatial sphere times time) can be related to Euclidean CFTs on \mathbb{R}^d in the following way. Consider, in the Euclidean CFT, the quantization on the foliation of concentric spheres around the origin, parameterized by their radius r . The quantum states are then defined on each of the $S^{d-1}(r)$ and translations in “time” are generated by dilatations. The conformal map¹¹ $r \rightarrow e^\tau$ maps the radial direction in flat space

¹¹ The flat space metric is indeed a multiple of the metric on the cylinder, at each point separately,

$$ds^2 = dr^2 + r^2 d\Omega_{d-1}^2 = (d\tau^2 + d\Omega_{d-1}^2)/\tau^2 . \quad (2.41)$$

onto non-compact direction τ of a cylinder $\mathbb{R} \times S^{d-1}$. For conformally invariant theories, *radial quantization* on flat space is therefore equivalent to quantization on the cylinder with Euclidean time τ . Primary operators on the plane are mapped to primary operators on the cylinder $\mathcal{O}_{\text{cyl}}(\tau, \theta^a) = e^{\Delta\tau} \mathcal{O}_{\mathbb{R}}(e^\tau \cdot \omega^i)$, with ω^i given in (2.6).

Euclidean theories obtained as the Wick rotation of a Lorentzian quantum field theory in this way, inherit the inner product and associated notion of Hermiticity. Real operators on the Euclidean cylinder map to operators satisfying

$$\mathcal{O}_{\mathbb{R}}(x)^\dagger = x^{-2\Delta} \mathcal{O}_{\mathbb{R}}\left(\frac{x^i}{x^2}\right), \quad (2.42)$$

on flat space. The analytic continuation from Lorentzian QFT implies the following Hermiticity properties for the conformal charges [50]

$$\hat{M}_{ij}^\dagger = -\hat{M}_{ij}, \quad \hat{D}^\dagger = \hat{D}, \quad \hat{P}_i^\dagger = \hat{K}_i. \quad (2.43)$$

This implies a number of “unitarity bounds” for Euclidean CFTs obtained this way. For example, real scalar primary operators \mathcal{O} satisfy

$$0 \leq \langle \mathcal{O} \hat{K}_i \hat{P}_j \mathcal{O} \rangle = \langle \mathcal{O} [\hat{K}_i, \hat{P}_j] \mathcal{O} \rangle = 2\Delta \delta_{ij} \langle \mathcal{O} \mathcal{O} \rangle. \quad (2.44)$$

Stronger constraints can be derived by considering more general two-point functions: $\Delta = 0$ or $\Delta \geq \frac{d}{2} - 1$ for scalar primaries and $\Delta \geq s + d - 2$ for primaries of spin s . Furthermore, under some technical assumption, it can be shown [50] that in these theories, \hat{D} is indeed diagonalizable with real eigenvalues and that every local operator can be written as a linear combination of primary operators.

2.4 The AdS/CFT correspondence

The ideas of holography were given explicit form in [4]. It was observed that contained within certain Lorentzian CFTs, there are theories of AdS supergravity in a spacetime of one more dimension than the CFT manifold. In the specific model analyzed in [4], the CFT is a version of Yang-Mills theory with gauge group $SU(N)$, that is conformally invariant¹² in $3 + 1$ dimensions, and which has gauge coupling constant g_{YM}^2 . In the limit of large N , and with the coupling

¹² In fact, the CFT in [4] is invariant under the $\mathcal{N} = 4$ superconformal group. Although supersymmetry was essential in the original derivation, we will not make explicit use of it here.

constant tuned so that the $\lambda \equiv g_{\text{YM}}^2 N$ is large but finite, this theory describes a classical supergravity theory in asymptotically $\text{AdS}_5 \times \text{S}^5$ spacetimes. This duality is conjectured to hold also when λ is not large, but with string theory on the gravity side of the duality. The string length is then proportional to the AdS length divided by $\lambda^{1/4}$. The string coupling constant g_s equals the gauge coupling constant g_{YM}^2 .

The observations made in [4], lead to the more general conjecture in [5, 6, 7, 58] that the large N limits of certain d -dimensional CFTs are equivalent to supergravity theories on AdS_{d+1} . The somewhat abstract “equivalence” of these two theories was made more precise. A relation between the operators in the CFT and the fields in AdS was proposed: an AdS field with spin s and mass m^2 is related to a primary operator in the CFT with the same spin s and conformal weight given by

$$(\Delta + s)(\Delta + s - d) = m^2 . \quad (2.45)$$

Furthermore, a rule to obtain CFT correlation functions from supergravity calculations was proposed: they are encoded in the dependence of the on-shell supergravity action on the asymptotic behavior of the fields at spatial infinity. More generally, the generating function of CFT correlation functions is computed by the supergravity action evaluated on the classical solution

$$Z_{\text{CFT}}[\phi_0] = Z_{\text{Sugra}}[\phi] . \quad (2.46)$$

These, and many more relations between quantities in the CFT and in AdS make up the so-called Witten *dictionary*.

It is possible to extend this duality to non-vanishing temperatures. When the CFT is in a thermal state, the dual description contains a black hole in AdS. Unlike flat space and de Sitter space, the “test-tube” like nature of AdS allows black holes to exist in thermal equilibrium with the rest of the spacetime (which at least contains a gas of gravitons, if no other matter fields). It was first discovered that AdS black holes exhibit rich thermodynamics in [59]. In particular, it was shown that “large” Schwarzschild anti-de Sitter black holes are thermodynamically favored with respect to Anti-de Sitter space above a certain temperature. This phase transition is called the Hawking-Page phase transition. This topic has received a new surge of attention in the context of holography: via AdS/CFT black hole phase transitions can be used to describe field theoretic phenomena at strong coupling, as done for example [23, 24, 25].

More generally, out-of-equilibrium processes have also been described in AdS/CFT. An example that provides motivation for the analysis in the next chapter is the holographic vitrification proposal of [26]. The authors modeled a super-cooled state in the CFT as a composite black hole system, a “multi-centered black hole”, in AdS. These configurations are characterized by both

disorder and rigidity: they are glass-like states. In the next chapter, we study a four-dimensional theory of AdS supergravity which can be embedded in M-theory. We find novel black hole solutions which are characterized by the presence of non-trivial matter field profiles outside of the black hole horizon.

Chapter 3

Black holes with halos [1]

This chapter is a reprint of [1], where we presented new AdS_4 black hole solutions in $\mathcal{N} = 2$ gauged supergravity coupled to vector and hypermultiplets. We focused on a particular consistent truncation of M-theory on the homogeneous Sasaki-Einstein seven-manifold M^{111} , characterized by the presence of one Betti vector multiplet. We numerically constructed static and spherically symmetric black holes with electric and magnetic charges, corresponding to M2 and M5 branes wrapping non-contractible cycles of the internal manifold. These configurations have nonzero temperature and are moreover surrounded by a massive vector field halo. We verified the first law of black hole mechanics and analyzed the thermodynamics and phase transitions in the canonical ensemble, interpreting the process in the corresponding dual field theory.

3.1 Introduction

The analysis of Anti-de Sitter (AdS) black hole solutions in theories of four-dimensional gauged supergravity is important for at least two reasons. On one hand, the [AdS/CFT](#) correspondence sheds light on the microstate structure of the supersymmetric configurations. In this regard, some recent developments [60, 61] successfully matched the [BPS](#) black hole [62] entropy with the ground state degeneracy of the corresponding twisted [ABJM](#) [9] theory, via supersymmetric localization. On the other hand, AdS black holes from string theory provide interesting gravitational backgrounds for top-down holographic approaches: one can map the rich thermodynamics and phase transitions of these systems to

processes in the dual field theory, providing a description of strongly coupled field theoretical phenomena, such as superconductivity [23, 24, 25].

The characterization and construction of solutions of gauged supergravity models coming from M-string theory is an important step in this direction. So far, much of the effort has been directed towards the analysis and characterization of black hole solutions of $\mathcal{N} = 2$ Abelian Fayet-Iliopoulos gauged supergravity [63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74]. The first example of static supersymmetric AdS_4 black holes was analytically constructed in [62], while previous studies [75] yielded naked singularities. In this model, the scalars are uncharged under the gauge group and solution-generating techniques of ungauged supergravity can be used to construct new configurations (see for instance [76, 77, 78, 79]).

The construction of analytic black hole solutions in other models of gauged supergravity, in particular those including hypermultiplets, initiated in [67, 80, 81], revealed to be much harder since the matter content includes charged scalars and massive vectors. Charged scalars and massive vectors are a generic feature of $\text{AdS}_4 \times M^6$ compactifications dual to ABJM theory. In these models a linear combination of the $U(1)$ gauge fields obtained from the reduction of the RR fields becomes massive due to the Higgs mechanism. This was shown in the original example for the compactification on CP^3 [9], and the Higgsing also occurs in other models (see for instance [82, 83]) arising from compactifications of M-theory on 7d Sasaki-Einstein manifold.

The fact that a $U(1)$ is Higgsed has nontrivial consequences for black hole physics, and in particular for the analysis of black hole bound states in AdS_4 [26]. A prerequisite for the existence of multi-centered black holes is that the electromagnetic interaction balances the gravitational one. A massive vector field decays exponentially, rather than polynomially, and this generally modifies the conditions for a bound state to exist. Moreover, bound configurations with magnetic charges would come with strings attached [26] due to the Meissner effect. All these ingredients can in principle play an important role in the existence and stability of these bound states.

The aim of this paper is take the first steps to address these problems, by constructing AdS_4 thermal black holes with an embedding in M-theory, surrounded by massive vectors and charged scalars. These solutions will provide suitable thermal backgrounds for the subsequent study of the probe stability.

We focus our attention on reductions of eleven dimensional supergravity whose vacua preserve $\mathcal{N} = 2$ supersymmetry. Such consistent truncations of M-theory on homogeneous seven-dimensional Sasaki-Einstein manifolds with $SU(3)$ structure were found in [83]. We work with a specific reduction of

11d supergravity on the SE_7 manifold M^{111} , with field theory dual in the class of [84, 85]. This truncation has a massive vector in its spectrum, which corresponds to a broken global symmetry in the dual field theory. Furthermore, it is characterized by the presence one Betti vector multiplet, dual to a global baryonic symmetry. On the gravity side, this multiplet contains light degrees of freedom, in particular massless vectors and scalars with mass $m^2 l^2 = -2$.

Zero-temperature, 1/4 BPS black hole solutions for various models, including Q^{111} and M^{111} , were found in [86], in the form of flows from AdS_4 to $AdS_2 \times S^2$ near-horizon geometries, by solving the BPS equations. Solutions of the same models, with planar horizons were previously obtained in [87, 88]. The presence of charged scalars considerably complicates the equations, hence the flows were obtained mostly numerically.

The black holes we present here correspond to nonzero temperature generalizations of the black holes of [86] and are found by solving the Einstein, Maxwell and scalar equations of motion. This reduces to a boundary value problem for a system of 14 coupled ODEs, which we solve numerically using a shooting method.

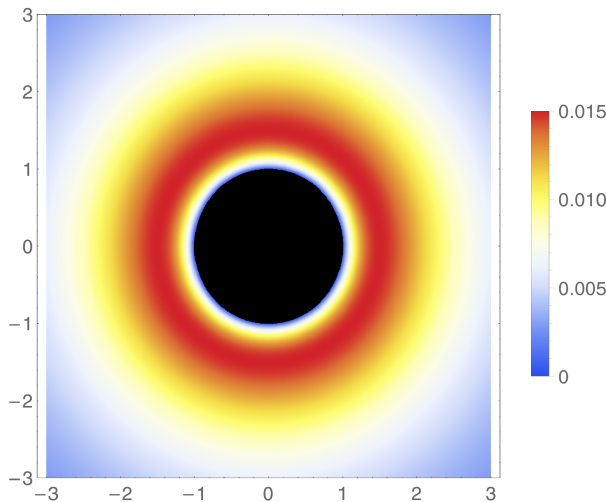


Figure 3.1: 2D plot of the massive vector field profile for a electric solution (details of the configuration are provided in Figure 3.2, Section 3.3.5). The massive vector profile is peaked outside the black hole, forming a “halo” or atmosphere surrounding the black hole.

We were able to construct dyonic AdS_4 black hole solutions with nontrivial

matter profiles outside the horizon¹. In particular, a massive vector field “halo” surrounds the black hole solutions, as depicted in Figure 3.1. The solutions asymptotically approach the AdS vacuum in which the vector remains massive, in contrast to the case of holographic superconductors [23, 24, 25], where a $U(1)$ symmetry is broken in the proximity of the horizon and is restored in the UV. We find that the presence of Betti vector multiplets is required in order to find (non-extremal) black hole solutions with nontrivial behavior of the massive vector. In the M-theory picture, these additional electric and magnetic charges correspond to wrapped M2 and M5-branes on cycles of the internal manifold.

To verify the accuracy of our numerics, we have checked that the first law of thermodynamics is satisfied on our solutions. We have performed holographic renormalization to compute the renormalized on shell action and subsequently studied the behavior of the free energy and its non-analytic points, searching for phase transitions.

The analysis of the stability of charged probe black holes in the background of these new configurations, along the lines of [91, 92, 93, 26], in view of the possible description of the holographic vitrification process is work in progress, and will be presented in a forthcoming paper. More directions in this regard will be presented in the outlook section.

3.2 Setup

3.2.1 Model M^{111}

The setup for our computations is the Abelian four-dimensional $\mathcal{N} = 2$ gauged supergravity theory obtained upon reduction of eleven-dimensional supergravity on the 7d Sasaki-Einstein manifold M^{111} . This is the coset manifold G/H where $G = SU(3) \times SU(2)$ and $H = SU(2) \times U(1)$. Its second Betti number is $b_2(M^{111}) = 1$, hence there is one nontrivial two-cycle around which M_2 branes can wrap. Therefore, the effective field theory contains one Betti vector multiplet in its spectrum. The same truncation can alternatively be obtained from the reduction on the seven-dimensional manifold Q^{111} (with $G = SU(2)^3$ and $H = U(1)^2$, and $b_2(Q^{111}) = 2$ hence two Betti multiplets), provided we consistently truncate one of the two Betti multiplets by suitably identifying two

¹The no-hair theorem of Bekenstein [34] rules out massive vector field hair in four-dimensional asymptotically flat spacetime. However, interactions among the different fields and AdS asymptotics are sufficient to evade the theorem. For further work on black holes and branes with massive vector fields, but Lifshitz asymptotics see [89]. Work on AdS black holes with massive vectors in $d > 4$ can be found in [90].

vectors and two scalar fields. The superconformal field theory dual to the Q^{111} model is the superconformal Chern-Simons flavored quiver of [84, 85] (see [94] for related work as well).

This theory admits an $\mathcal{N} = 2$ supersymmetric AdS vacuum². The field content of the theory is the gravity multiplet, two vector multiplets ($n_v = 2$) and the universal hypermultiplet. We essentially follow the conventions of [82, 83]³. The Lagrangian has the form

$$S = \int \frac{1}{2} R * 1 + g_{i\bar{j}} Dt^i \wedge * D\bar{t}^{\bar{j}} + h_{uv} Dq^u \wedge * Dq^v \quad (3.1)$$

$$+ \frac{1}{4} \text{Im} \mathcal{N}_{\Lambda\Sigma} F^\Lambda \wedge * F^\Sigma + \frac{1}{4} \text{Re} \mathcal{N}_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma - V, \quad (3.2)$$

where $t^i = \tau^i + ib^i$, ($i = 1, 2$) parameterize the two complex scalars in the vector multiplets and q^u , ($u = 1, \dots, 4$) those in the hypermultiplet. The vectors F^Λ , ($\Lambda = 0, 1, 2$) come from the two vector multiplets and the gravity multiplet. We work in the symplectic frame where all gaugings are electric⁴, and the model is characterized by the corresponding holomorphic prepotential

$$F(X) = -2i\sqrt{X^0(X^1)^2 X^2}. \quad (3.3)$$

The scalars in the vector multiplets parameterize the special Kähler manifold $\left(\frac{SU(1|1)}{U(1)}\right)^2$ with metric

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K}(z, \bar{z}), \quad \mathcal{K} = -\log[i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)] . \quad (3.4)$$

where $F_\Lambda = \partial_\Lambda F$. From the covariantly holomorphic sections (X^Λ, F_Λ) we define moreover the sections

$$(L^\Lambda, M_\Lambda) = e^{\mathcal{K}/2} (X^\Lambda, F_\Lambda) . \quad (3.5)$$

We choose sections such that $X^\Lambda = \{X^0, X^1, X^2\} = \{1, t_1^2, t_2^2\}$. The period matrix $\mathcal{N}_{\Lambda\Sigma}$ encodes the (scalar dependent) kinetic terms for the vector fields, and it is obtained via the special geometry relation

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i \frac{\text{Im} F_{\Lambda\Delta} \text{Im} F_{\Sigma\Gamma} X^\Delta X^\Gamma}{\text{Im} F_{\Delta\Gamma} X^\Delta X^\Gamma}, \quad (3.6)$$

² See for instance [95] for further models of gauged $\mathcal{N} = 2$ supergravity coupled to hypermultiplets with fully supersymmetric vacua.

³ In the original paper [82] the vector kinetic terms have a factor 1/2 instead of 1/4 in front. However, their definition of \mathcal{N}_{IJ} includes a factor 1/2 with respect to ours, hence the total factor 1/4 in our Lagrangian. These conventions differ with respect to those of [96] and [86] by the following: $A_{\text{here}} = \sqrt{2} A_{\text{there}}$ and $k_\Lambda^u = \frac{1}{\sqrt{2}} k_\Lambda^u$, as already noticed (see footnote (10) of [83]).

⁴ This is the four-dimensional theory obtained upon reduction, after dualization of the massive tensor multiplet in a massive vector multiplet (full details in [82]).

where $F_{\Delta\Sigma} = \frac{\partial F}{\partial X^\Delta X^\Sigma}$. Its explicit form is reported in the Appendix of [1].

The universal hypermultiplet contains the 4 hyperscalars $q^u = (\phi, a, \xi, \bar{\xi})$, which parameterize the quaternionic Kähler manifold $\frac{SU(2,1)}{S(U(2) \times U(1))}$ with metric h_{uv} of the form

$$h_{uv} dq^u dq^v = d\phi^2 + \frac{e^{4\phi}}{4} \left[da - \frac{i}{4} (\xi d\bar{\xi} - \bar{\xi} d\xi) \right]^2 + \frac{e^{2\phi}}{4} d\xi d\bar{\xi}. \quad (3.7)$$

This quaternionic Kähler manifold has constant negative curvature $R_q = -24 = -8n_h(2 + n_h)$ [97], where n_h is the number of hypermultiplets in the theory (in our case $n_h = 1$).

The covariant derivatives for the vector multiplets and the hyperscalars are given by

$$Dt^i = dt^i + k_\Lambda^i A^\Lambda, \quad Dq^u = dq^u + k_\Lambda^u A^\Lambda, \quad (3.8)$$

where k_Λ^i and k_Λ^u are the Killing vectors corresponding to the gauging of the special Kähler and the quaternionic manifold respectively. The quaternionic Killing vectors k_Λ^u can be derived from the Killing prepotentials P_Λ^x which satisfy the relation $\Omega_{vw}^x k_\Lambda^w = -\nabla_v P_\Lambda^x$ [98, 96, 99], where $\Omega_{vw}^x = d\omega^x + \frac{1}{2} \epsilon^{xyz} \omega^y \wedge \omega^z$ is the curvature on the quaternionic manifold. In the model we consider, only a $U(1)$ isometry of the hypermultiplet manifold is gauged. Thus, the covariant derivatives for the vector multiplet scalars boil down to simple derivatives, as $k_\Lambda^i = 0$. The hyperscalars are charged, however. The prepotentials and Killing vectors of the gauging are [82, 83]:

$$P_0 = 6P_a - 4P_\xi, \quad P_1 = 4P_a, \quad P_2 = 2P_a, \quad (3.9)$$

where

$$P_a = \begin{pmatrix} \frac{ie^{2\phi}}{4} & 0 \\ 0 & -\frac{ie^{2\phi}}{4} \end{pmatrix}, \quad P_\xi = \begin{pmatrix} \frac{i}{2}(1 - \xi\bar{\xi}e^{-2\phi}) & -i\xi e^{-\phi} \\ -i\bar{\xi}e^{-\phi} & -\frac{i}{2}(1 - \xi\bar{\xi}e^{-2\phi}) \end{pmatrix}, \quad (3.10)$$

and $P_\Lambda = P_\Lambda^x (-\frac{i}{2}\sigma^x)$. Therefore,

$$k_0 = -6\partial_a + 4i(\xi\partial_\xi - \bar{\xi}\partial_{\bar{\xi}}), \quad k_1 = -4\partial_a, \quad k_2 = -2\partial_a. \quad (3.11)$$

Finally, the scalar potential of the theory, which couples scalars in the vector multiplets and hyperscalars, is given by

$$V(t, \bar{t}, q) = (g_{i\bar{j}} k_\Lambda^i k_\Sigma^{\bar{j}} + 4h_{uv} k_\Lambda^u k_\Sigma^v) \bar{L}^\Lambda L^\Sigma + (f_i^\Lambda f_{\bar{j}}^\Sigma g^{i\bar{j}} - 3\bar{L}^\Lambda L^\Sigma) P_\Sigma^x P_\Lambda^x. \quad (3.12)$$

where L^Λ are defined in (3.5) and $f_i^\Lambda = (\partial_i + \frac{1}{2}\partial_i \mathcal{K})L^\Lambda$.

Given this specific form of the gauging in the M^{11} truncation, one of the vectors becomes massive via the Higgs mechanism. The spectrum then contains (see Table 7 of [83])

- the gravity multiplet, containing the metric $g_{\mu\nu}$ and a massless vector,
- a Betti vector multiplet, containing the massless vector and a complex scalar (two real fields) of mass $m^2 l^2 = -2$ (in our conventions, the Breitenlohner-Freedman bound is $m^2 l^2 \geq -9/4$), each with $\Delta = (2, 1)$,
- a massive vector multiplet, containing a massive vector of mass $m^2 l^2 = 12$ (which corresponds holographically to a vector operator with weight $\Delta = 5$), which has eaten its axion a and five scalars of mass $m^2 l^2 = (18, 10, 10, 10, 4)$ corresponding to $\Delta = (6, 5, 5, 5, 4)$.

Before proceeding further, let us remind the reader about the asymptotic fall-off of vectors and scalars in AdS_4 spacetime. The scaling dimension of an operator dual to a massive p -form in AdS_4 spacetime is given by the formula [58]

$$\Delta_{\pm} = \frac{3}{2} \pm \frac{1}{2} \sqrt{(3-2p)^2 + 4m^2 l^2} . \quad (3.13)$$

A vector field ($p = 1$) dual to an operator of scaling dimension Δ behaves as (r is the AdS radial coordinate, and the boundary is reached at $r \rightarrow \infty$)

$$r^{-2+\Delta_+} \quad \text{and} \quad r^{1-\Delta_+} . \quad (3.14)$$

A scalar field ($p = 0$) instead behaves as

$$r^{-3+\Delta_+} \quad \text{and} \quad r^{-\Delta_+} . \quad (3.15)$$

We will come back to these asymptotic fall-offs later on when dealing with the explicit AdS_4 solutions.

3.2.2 Consistent truncation

In finding black hole solutions we will make a simplifying assumption: we retain only *one* hyperscalar. Indeed one can see that the complex hyperscalar ξ can be consistently truncated away, and the field a is the Stueckelberg field which can be consistently set to the value zero by a choice of gauge. Our truncated theory will then be characterized by the following matter content: two massless vector fields, a massive one, and five scalars of masses $m^2 l^2 = (18, 10, 4, -2, 2)$ which correspond to dual operators of dimensions $\Delta = (6, 5, 4, (2, 1), (2, 1))$ where $(2, 1)$ indicates the two normalizable modes for a scalar with mass $m^2 l^2 = -2$.

Given this truncation, the only nonvanishing components of the quaternionic Killing prepotentials are

$$P_{\Lambda}^3 = (4 - 3e^{2\phi}, -2e^{2\phi}, -e^{2\phi}) , \quad (3.16)$$

hence the Killing vectors appearing in the gaugings (3.11) are

$$k_{\Lambda}^a = -(6, 4, 2) . \quad (3.17)$$

In order to simplify our computation, we can assume a specific value for the Freund–Rubin parameter appearing in [86], $e_0 = 6$, which leads to the fixed value of AdS radius $l = \frac{1}{2} \left(\frac{e_0}{6}\right)^{3/4} = 1/2$ – see for instance formula (3.16) of [86].

Putting all gauging data together, and redefining the hypermultiplet field $\phi = \log \sigma$, the action (3.1) can be rewritten in the form

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} R - V \right) + S_V + S_H , \quad (3.18)$$

where the scalar potential is, using (3.12),

$$\begin{aligned} V = \sigma^4 & \left(\frac{(2b_1b_2 + b_1^2 + 3)^2}{\tau_1^2\tau_2} + \frac{2(b_1 + b_2)^2}{\tau_2} + \frac{4\tau_2b_1^2}{\tau_1^2} + \frac{\tau_1^2}{\tau_2} + 2\tau_2 \right) \\ & - 8\sigma^2 \left(\frac{2}{\tau_1} + \frac{1}{\tau_2} \right) . \end{aligned} \quad (3.19)$$

It has an AdS minimum $V_{\min} = -12$ for the following values of the scalar fields

$$\tau_1 = \tau_2 = \sigma = 1 , \quad b_1 = b_2 = 0 . \quad (3.20)$$

The action for the hypermultiplet sector is

$$S_H = -\frac{1}{2} \int d^4x \sqrt{-g} \left[2(\nabla \log \sigma)^2 + \frac{1}{2} \sigma^4 (\nabla a - (6A_0 + 4A_1 + 2A_2))^2 \right] , \quad (3.21)$$

where we can see that the scalar field a acts as a Stueckelberg field responsible for the Higgsing of the linear combination $6A_0 + 4A_1 + 2A_2$. Finally, the vector multiplet Lagrangian reads

$$\begin{aligned} S_V = \frac{1}{4} \int d^4x \sqrt{-g} & \left[-2(\nabla(\log \tau_1))^2 - (\nabla(\log \tau_2))^2 - \frac{2(\nabla b_1)^2}{\tau_1^2} - \frac{(\nabla b_2)^2}{\tau_2^2} \right] \\ & + \frac{1}{4} \int (\text{Im} \mathcal{N}_{\Lambda\Sigma} F^{\Lambda} \wedge *F^{\Sigma} + \text{Re} \mathcal{N}_{\Lambda\Sigma} F^{\Lambda} \wedge F^{\Sigma}) , \end{aligned} \quad (3.22)$$

with \mathcal{N} given in (3.6). The supergravity vector fields A^{Λ} can be expressed as well as linear combination of the massless eigenstates $\mathcal{A}_1, \mathcal{A}_2$ and the massive

one \mathcal{B} , in this way

$$\begin{aligned} A^0 &= \frac{1}{2}\mathcal{A}_1 + \frac{\sqrt{3}}{2}\mathcal{B} , \\ A^1 &= -\frac{1}{2}\mathcal{A}_1 + \frac{\sqrt{3}}{6}\mathcal{B} - \frac{1}{\sqrt{6}}\mathcal{A}_2 , \\ A^2 &= -\frac{1}{2}\mathcal{A}_1 + \frac{\sqrt{3}}{6}\mathcal{B} + \frac{2}{\sqrt{6}}\mathcal{A}_2 . \end{aligned} \tag{3.23}$$

We verified that this action reduces to that of [82] if we identify $t_1 = t_2$ and $A^1 = A^2$. These identifications correspond to switching off the Betti vector multiplet, which contains in particular the massless vector \mathcal{A}_2 . The universal SE^7 reduction of [82] coincides with the truncation on $S^7 = SU(4)/SU(3)$ that retains the $SU(4)$ left-invariant modes.

3.3 Finding black hole solutions

3.3.1 Static black hole ansatz

We focus on the search for static and spherically symmetric solutions of the form⁵

$$ds^2 = -e^{-\beta(r)}h(r)dt^2 + \frac{dr^2}{h(r)} + r^2 d\Omega^2 , \tag{3.24}$$

which allows for asymptotically locally AdS spacetimes. The five scalar fields have only radial dependence:

$$\tau_1 = \tau_1(r) , \quad \tau_2 = \tau_2(r) , \quad b_1 = b_1(r) , \quad b_2 = b_2(r) , \quad \sigma = \sigma(r) . \tag{3.25}$$

For the vectors, we choose an ansatz that can describe the fields around a static black hole with both electric and magnetic charge,

$$\begin{aligned} \mathcal{A}_{1,t} &= \xi_1(r) , & \mathcal{A}_{2,t} &= \xi_2(r) , & \mathcal{B}_t &= \zeta(r) , \\ \mathcal{A}_{1,\varphi} &= P^1 \cos \theta , & \mathcal{A}_{2,\varphi} &= P^2 \cos \theta , & \mathcal{B}_\varphi &= P^m \cos \theta . \end{aligned} \tag{3.26}$$

⁵ We look for configuration of spherical horizon topology but we expect that solution with flat or hyperbolic event horizons exist as well, as BPS solutions of this kind were found in [86, 87].

More precisely, the charges are the integral of the flux of the field strength $F_{\mu\nu}$ and its dual $G_{\mu\nu}$ through the sphere at spatial infinity:

$$Q_i = \frac{1}{4\pi} \int_{S_\infty^2} G_{\mathcal{A}_i} , \quad P^i = \frac{1}{4\pi} \int_{S_\infty^2} F_{\mathcal{A}_i} , \quad (3.27)$$

with the dual defined as

$$G_{\mu\nu,\Lambda} = \frac{1}{4} \sqrt{-g} \epsilon_{\mu\nu}{}^{\rho\sigma} \frac{\partial \mathcal{L}}{\partial F^{\rho\sigma,\Lambda}} . \quad (3.28)$$

The equations of motion derived from (3.1) with the above ansatz are given in Appendix B of [1]. In total, there are 14 degrees of freedom: the equations of motion for the metric components β and h are first order, yielding one dynamic component each. The scalars τ_1 , τ_2 , b_1 , b_2 and σ on the other hand, have second order equations of motion. Just like the massive vector mode ζ . Due to charge conservation and gauge invariance, there are no dynamical components that correspond to the massless vectors ξ_1 and ξ_2 .

In the duality frame we consider, the hypermultiplets and the gravitini are electrically charged. Therefore, the following Dirac quantization conditions need to hold:

$$P^\Lambda k_\Lambda^u(\bar{q}) \in \mathbb{Z} , \quad P^\Lambda P_\Lambda^3(\bar{q}) \in \mathbb{Z} , \quad (3.29)$$

where $P_\Lambda^3(\bar{q}) = \{1, -2, -1\}$ and $k_\Lambda^u(\bar{q}) = -\{6, 4, 2\}$ are respectively the Quaternionic Killing prepotentials and Killing vectors computed on the vacuum solution (3.20). The first Dirac quantization condition in (3.29) is automatically satisfied on shell for the particular assumptions on the ansatz we made, while the second condition in (3.29), taking into account (3.23), reads

$$2P^1 \in \mathbb{Z} . \quad (3.30)$$

Furthermore, the Maxwell equation imposes to the condition

$$P^\Lambda k_\Lambda^u = 0 \quad (3.31)$$

along the entire flow. In our case this means that the massive vector in (3.26) field has zero magnetic component:

$$P^m = 0 . \quad (3.32)$$

Releasing the condition of spherical symmetry would allow for a nontrivial magnetic component. In particular, this would result in vortex lines of the Nielsen-Olsen [100] type⁶.

⁶ The strings stretched between probes mentioned in the introduction would manifest themselves as vortex-type solutions in this kind of truncation. This would be interesting to study, but it goes beyond the scope of the present work. We hope to come back to this point in the future.

Similarly to [101, 102], the equation of motion and the background fields have the following scaling symmetry,

$$t \rightarrow \gamma t, \quad \beta \rightarrow \beta + 2 \log \gamma, \quad \xi_1 \rightarrow \frac{\xi_1}{\gamma}, \quad \xi_2 \rightarrow \frac{\xi_2}{\gamma}, \quad \zeta \rightarrow \frac{\zeta}{\gamma}, \quad (3.33)$$

which can be used to choose without loss of generality the asymptotic value of the metric function β at infinity. Indeed in what follows we will choose

$$\lim_{r \rightarrow \infty} \beta = 0. \quad (3.34)$$

The black hole solutions are most conveniently represented by the coordinate u , which is related to the radial Schwarzschild coordinate as

$$u = \log \left(\frac{r}{r_H} \right), \quad (3.35)$$

where r_H is the location of the event horizon. The horizon is retrieved by the $u = 0$ limit, while asymptotically $u \rightarrow \infty$ the solution approaches AdS₄ spacetime, with radius $l_{\text{AdS}} = 2$, which is kept fixed in our computations⁷. In these new coordinate u , the metric reads:

$$ds^2 = -e^{2u-\beta(u)} r_H^2 H(u) dt^2 + \frac{du^2}{H(u)} + e^{2u} r_H^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (3.36)$$

where we defined

$$h(u) = r_H^2 e^{2u} H(u). \quad (3.37)$$

As an elementary consistency check, the dyonic Reissner-Nordström solution is obtained by setting all the scalar fields at their vacuum value (3.20) throughout the entire flow. The solution is then characterized by the following warp factors:

$$\beta(u) = 0, \quad (3.38)$$

$$H(u) = 4 + \frac{e^{-2u}}{r_H^2} - \frac{(16r_H^4 + 4r_H^2 + (P^1)^2 + Q_1^2)e^{-3u}}{4r_H^4} + \frac{((P^1)^2 + Q_1^2)e^{-4u}}{4r_H^4},$$

with the additional conditions

$$P^2 = 0, \quad Q_2 = 0, \quad (3.39)$$

⁷ It is nevertheless straightforward to reinstate the gauge coupling constant in the action (3.1), allowing for a different value of the cosmological constant and AdS radius. The authors of [101, 102] moreover find another scaling symmetry which allows to pick $r_H = 1$ without loss of generality. This is due to the fact that they deal with planar horizons – in case of spherical horizons such additional scaling symmetry (3.23) of [102] is absent.

coming from the scalar equations of motion. The Reissner-Nordström solution in (3.38) is parameterized by the electromagnetic charges Q_1 and P^1 and the radius of the event horizon r_H which can be equivalently traded for the mass M of the black hole.

3.3.2 Strategy for numeric simulations

We will use numerical tools to solve the equations of motion subject to the relevant boundary conditions. This allows us to find smooth configurations which in the UV approach AdS spacetime and in the IR form the black hole horizon. In order to preserve the AdS_4 asymptotics we need to set the diverging modes of the heavy scalars and of the massive vector field to zero (see eq. (3.14)-(3.15)). The requirement of regularity on the black hole horizon will relate the derivative of the scalar fields to their values at the horizon.

To find solutions interpolating between AdS_4 and the black hole horizon, we will use a shooting method. In a first step, we provide boundary conditions in the IR, i.e. on the black hole horizon at $u = 0$, and integrate the equations of motion towards the boundary of AdS. Secondly and independently, we choose boundary conditions in the UV (at a value $u \gg 1$, so $r \gg l$) and integrate the equations into the IR. At some intermediate point in the bulk (for example $u = 1$), we obtain two values for each of the fields, depending nonlinearly on both sets of boundary conditions we chose. We then employ an optimization algorithm to minimize the difference and finally obtain the matching, by tweaking the boundary conditions on both the black hole horizon and the asymptotic boundary of AdS.

As mentioned before, there are 14 dynamic degrees of freedom. As we will see below, we can tune 16 boundary conditions for the fields⁸, as well as the value of r_H and four black hole charges (two electric and two magnetic). We therefore expect to find a 7-parameter family of solutions⁹.

⁸ It turns out that for each field except for the scalars with $m^2 l^2 = -2$, the conditions of a smooth black hole horizon and asymptotically AdS fix as many boundary conditions as there are degrees of freedom. The light scalars have two normalizable modes for $r \rightarrow \infty$, both of which are compatible with the asymptotic AdS behavior.

⁹ Due to the nonlinear nature of this system, this naive expectation is possibly incorrect. In principle, there might be no solutions at all, or there could be multiple 7-parameter families of solutions, up to a countable infinite number of them.

3.3.3 Asymptotic behavior of the fields

The solution near the AdS boundary is characterized by the fall-off of the fields. They are most conveniently expressed in terms of the coordinate $z = 1/r = e^{-u}/r_H$. The most general boundary conditions can be derived by considering the equations of motion order by order in z , starting from the leading terms which are fixed by the requirement of asymptotic AdS. To obtain the most general solution, one should take into account terms of the form z^n as well as $z^n \log(z)^m$ (see for example [86]). The equations of motion will require most (but not all) of the logarithmic terms to vanish. In total, we find 9 coefficients ($h_{(3)}, \tau_{(1)}, b_{(1)}, \tau_{(2)}, b_{(2)}, \sigma_{(4)}, b_{(5)}, \sigma_{(6)}, \zeta_{(4)}$), which encode the asymptotic behavior as follows.

The components of the metric have the following fall-off:

$$H = 4 + \left(1 + 6\tau_{(1)}^2 + 6b_{(1)}^2\right) z^2 + h_{(3)} z^3 + \mathcal{O}(z^4) , \quad (3.40a)$$

$$\beta = \frac{3}{2} \left(\tau_{(1)}^2 + b_{(1)}^2\right) z^2 + \frac{4}{5} \left(\tau_{(1)}^3 - \tau_{(1)} b_{(1)}^2 + 5\tau_{(1)} \tau_{(2)} + 5b_{(1)} b_{(2)}\right) z^3 + \mathcal{O}(z^4) ,$$

where $\tau_{(1)}$ and $b_{(1)}$ are the leading fall-off coefficients of the lightest scalar fields (see below). If they vanish, we recover the familiar AdS-Reissner-Nordström with $M = -h_{(3)}/2$ as in (3.38). As mentioned before, we choose the time coordinate such that $\beta|_{z=0} = 0$.

The behavior of the scalar fields can be expressed as a power series in $z \sim 0$ as well (for the sake of clarity, we omit terms that are at least quadratic in the coefficients)

$$\tau_1 = 1 + \tau_{(1)} z + \tau_{(2)} z^2 + \dots + \left(\frac{4}{3} \sigma_{(4)} - \frac{1}{12} \tau_{(2)} + \dots\right) z^4 + \dots$$

$$- \left(\sigma_{(6)} + \frac{1}{2} \sigma_{(4)} + \frac{1}{80} \tau_{(2)} + \dots\right) z^6 + \mathcal{O}(z^7) ,$$

$$\tau_2 = 1 - 2\tau_{(1)} z - (2\tau_{(2)} + \dots) z^2 + \dots + \left(\frac{4}{3} \sigma_{(4)} + \frac{1}{6} \tau_{(2)} + \dots\right) z^4$$

$$+ \dots - \left(\sigma_{(6)} + \frac{1}{2} \sigma_{(4)} + \frac{1}{40} \tau_{(2)} + \dots\right) z^6 + \mathcal{O}(z^7) ,$$

$$b_1 = b_{(1)} z + b_{(2)} z^2 + \dots - \left(\frac{1}{12} b_{(2)} + \dots\right) z^4 + (b_{(5)} + \dots) z^5 + \mathcal{O}(z^6) ,$$

$$b_2 = -2b_{(1)}z - (2b_{(2)} + \dots)z^2 + \dots + \left(\frac{1}{6}b_{(2)} + \dots\right) + (b_{(5)} + \dots)z^5 + \mathcal{O}(z^6) ,$$

$$\sigma = 1 + \dots + (\sigma_{(4)} + \dots)z^4 + \dots + (\sigma_{(6)} + \dots)z^6 + \mathcal{O}(z^7) . \quad (3.40b)$$

To zeroth order in z , the scalars are in the AdS extremum of the potential (3.20). As anticipated in Section 3.2.2, the excitations around this minimum are characterized by the eigenvalues $m^2 l^2 = (18, 10, 4, -2, -2)$ of the mass matrix. Therefore, there are two independent components of the fields with fall-off z (corresponding to a $\Delta = 1$ source or operator in the CFT, depending on the quantization scheme), parameterized by $\tau_{(1)}$ and $b_{(1)}$; there are two modes falling off like z^2 , proportional to $\tau_{(2)}$ and $b_{(2)}$; and there are single modes proportional to z^4 , z^5 and z^6 , parameterized by $\sigma_{(4)}$, $b_{(5)}$ and $\sigma_{(6)}$, respectively. Furthermore, interactions give rise to terms quadratic in these coefficients, which are included in the “...”.

Finally, the massive vector field ζ has the following fall-off

$$\zeta = \frac{\sqrt{2}}{10} (Q_2 \tau_{(1)} - b_{(1)} P^2) z^2 + \frac{\sqrt{2}}{3} (Q_2 \tau_{(2)} - b_{(2)} P^2 + \dots) z^3$$

$$- \frac{3\sqrt{2}}{70} (Q_2 \tau_{(1)} - b_{(1)} P^2 + \dots) z^4 \log(z) + \zeta_{(4)} z^4 + \mathcal{O}(z^5) . \quad (3.40c)$$

The presence of the massive vector on the gravity side signals a broken global flavor symmetry in the dual field theory. The parameter $\zeta_{(4)}$ is related to the expectation value of a dual operator with dimension $\Delta = 5$.

3.3.4 Fields at the horizon

The boundary conditions on the black hole horizon, which in our conventions is located at $u = 0$, must ensure the existence of a smooth horizon. The timelike component of the metric $g_{tt} \propto H$ must vanish while none of the scalar fields must diverge. Furthermore, consistency of the equation of motion requires the massive vector field $\zeta(u)$ to vanish (leaving only its derivative as a free parameter) and determine the derivatives of the scalar fields in terms of their values at the horizon. All together, the fields near $u \approx 0$ are characterized by 7 parameters $(\beta^{(h)}, \sigma^{(h)}, \tau_1^{(h)}, \tau_2^{(h)}, b_1^{(h)}, b_2^{(h)}, \zeta'^{(h)})$,

$$H = u \left(12 + \frac{1}{r_H^2} + \frac{1}{12r_H^4} [-3(Q_1^2 + Q_2^2 + (P^1)^2 + (P^2)^2) \right.$$

$$\left. - (Q_2^2 - (P^2)^2)(4\tau_1^{(h)} - \tau_2^{(h)}) + 2Q_2 P^2(4b_1^{(h)} - b_2^{(h)}) \right]$$

$$\begin{aligned}
& + 2\sqrt{6}(Q_1 P^2 + Q_2 P^1)(b_1^{(h)} - b_2^{(h)}) \\
& + 2\sqrt{6}(Q_1 Q_2 - P^1 P^2)(\tau_1^{(h)} - \tau_2^{(h)}) \Big] + \dots \Big) + \mathcal{O}(u^2) , \\
\beta &= \beta^{(h)} + \mathcal{O}(u) , \quad \tau_1 = \tau_1^{(h)} + \mathcal{O}(u) , \quad \tau_2 = \tau_2^{(h)} + \mathcal{O}(u) , \quad (3.41) \\
b_1 &= b_1^{(h)} + \mathcal{O}(u) , \quad b_2 = b_2^{(h)} + \mathcal{O}(u) , \quad \sigma = \sigma^{(h)} + \mathcal{O}(u) , \\
\zeta &= \zeta'^{(h)} u + \mathcal{O}(u^2) .
\end{aligned}$$

3.3.5 Solutions: examples

At this point, there are 21 free parameters:

- 9 boundary conditions on the asymptotic boundary of AdS,
- 7 boundary conditions on the black hole horizon,
- the radius of the event horizon r_H ,
- the electromagnetic charges of the black hole (Q_1, Q_2, P^1, P^2) which represent conserved quantities of the two massless vector fields.

To obtain a consistent AdS black hole solution, however, one cannot choose all of these parameters arbitrarily. There are 14 constraints from the requirement that the IR solution to the equations of motion (integrated from the black hole horizon outward) evolve smoothly into the UV solution (integrated from the boundary of AdS inward). Indeed, the equations of motion are a system of 14 coupled first order ODEs. Thus, 14 integration constants must be fixed to ensure a smooth solution. We collectively denote them by q_{integr} .

The system is then still underdetermined: we have $21 - 14 = 7$ tunable parameters which are not fixed by the equations of motion, which by themselves specify each black hole solution taken into consideration. These are the four electromagnetic charges (Q_1, Q_2, P^1, P^2) , the leading modes of the light scalar fields $\tau_{(1)}$ and $b_{(1)}$, and the radius of the event horizon r_H . We denote these parameters by q_{input} .

With this in mind, one can find solutions numerically. We developed a Mathematica code that, given a set of external tunable parameters q_{input} , allows us to find black hole solutions by finding appropriate q_{integr} . The results are fully backreacted configurations representing thermal black hole solutions with nontrivial radial profile for the matter present in the theory.

We find electric, magnetic and dyonic solutions. The behavior of the fields as a function of the radial coordinate u is displayed in Figure 3.2 for the purely electric configuration, and in Figure 3.3 for the purely magnetic one. In the latter case the massive vector is zero (see discussion in Section 3.1).

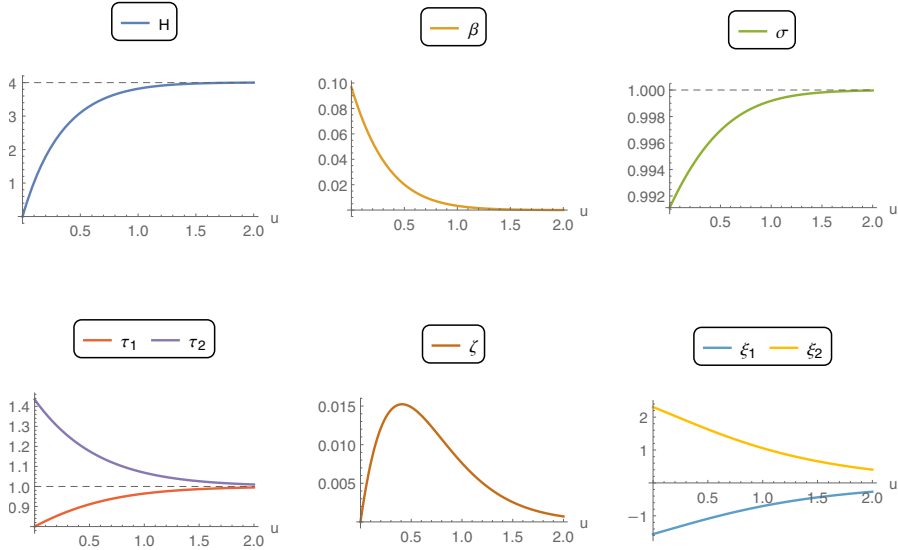


Figure 3.2: Purely electric black hole solution with $r_H = 1$, $Q_1 = 2$, $Q_2 = -3$, $P^1 = P^2 = 0 = \tau_{(1)} = b_{(1)}$. The integration constants obtained with the numerical shooting technique are (we report them here up to 3 digits) $\beta^{(h)} = 0.097$, $\sigma^{(h)} = 0.991$, $\tau_1^{(h)} = 0.800$, $\tau_2^{(h)} = 1.434$, $b_1^{(h)} = b_2^{(h)} = 0$, $\zeta'^{(h)} = 0.092$, $h_{(3)} = -7.816$, $\tau_{(1)} = 0$, $\tau_{(2)} = -0.270$, $b_{(1)} = b_{(2)} = b_{(5)} = 0$, $\sigma_4 = -0.036$, $\sigma_{(6)} = -0.082$, $\zeta_{(4)} = -0.728$. The IR solution was integrated from $u = 10^{-12}$ to $u = 1$, and the UV solution was integrated from $u = 10 \rightarrow u = 1$. We used 30 digits of numerical precision. The IR and UV solutions at $u = 1$ differ by $\sum_i (\Delta\varphi_i)^2 = 1.22 \cdot 10^{-23}$, where $\varphi = (H, \beta, \tau_1, \tau_1', \tau_2, \tau_2', b_1, b_1', b_2, b_2', \sigma, \sigma', \zeta, \zeta')$.

The vector condensate surrounding the black hole and is moreover visualized in the 2d radial the plot in Figure 3.1. Circles of radius r in the plot truthfully correspond to spheres with surface area $4\pi r^2$ in the AdS black hole geometry. However, radial distances in the plot are related to radial distances in the AdS black hole geometry by $dr_{\text{plot}} = dr_{\text{BH}}/\sqrt{H(r_{\text{BH}})}$, where H is given in Figure 3.2. The massive vector field ζ vanishes at the event horizon, and it is peaked at a finite radial value outside the black hole horizon. The field ζ

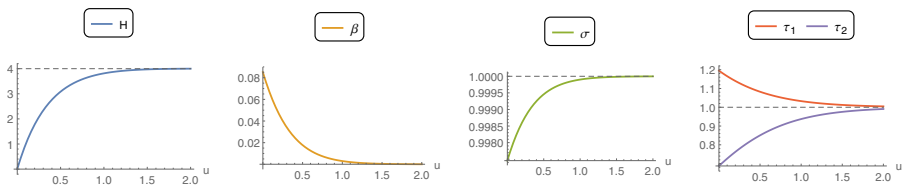


Figure 3.3: Purely magnetic black hole solution with $r_H = 1$, $P^1 = 2$, $P^2 = -3$, $Q^1 = Q^2 = 0 = \tau_{(1)} = b_{(1)}$. The massive vector ζ , as well as ξ_1 and ξ_2 vanish identically in this case. The integration constants are $\beta^{(h)} = 0.085$, $\sigma^{(h)} = 0.997$, $\tau_1^{(h)} = 1.195$, $\tau_2^{(h)} = 0.687$, $b_1^{(h)} = b_2^{(h)} = 0$, $\zeta'^{(h)} = 0$, $h_{(3)} = -7.850$, $\tau_{(1)} = 0$, $\tau_{(2)} = 0.257$, $b_{(1)} = b_{(2)} = b_{(5)} = 0$, $\sigma_4 = -0.004$, $\sigma_{(6)} = 0.0229$, $\zeta_{(4)} = 0$. We used 30 digits of numerical precision. The IR and UV solutions at $u = 1$ differ by $\sum_i (\Delta\varphi_i)^2 = 1.16 \cdot 10^{-23}$.

by itself is a massive object surrounding the black hole: the configuration can therefore be seen as an example of “composite” back-reacted configuration in AdS spacetime. The interactions dictated by the nontrivial couplings of the supergravity Lagrangian allow this massive object to gravitate outside the black hole horizon without falling in.

It would be interesting to understand more deeply why the massive vector halo is stable outside the horizon. For example, one might attempt to analyze the stability of a “probe” massive vector particle in this background, in analogy with the probe black hole calculation of [26]. However, the point particle approximation can be expected to break down since the de Broglie wavelength of such a particle is of the order of the AdS length scale. Furthermore, there is kinetic mixing between the vectors in the supergravity Lagrangian, which is expected to affect the effective particle interactions. One would need to overcome these obstacles to obtain the correct form of the effective potential for the probe, and determine its stability.

Before concluding, let us stress one difference between our solutions and those treated for instance in [23, 24, 25, 103]. In our case the configuration has a massive vector in the Kaluza-Klein spectrum. Therefore the related symmetry is broken already in the vacuum of the theory, and it is never restored. However, for the solutions in [23, 24, 25, 103] describing holographic superconductors, the vector field in the vacuum of the theory has zero mass, as one can see from the asymptotic expansion of the fields. The breaking of the $U(1)$ symmetry happens in the latter case only in the proximity of the horizon, while the symmetry is restored at the boundary. One can actually see that the linearized theory

considered for instance in [103] retains the \mathcal{A}_1 gauge field and the scalar mode ξ that we instead truncated away.

The configurations we find are also different from those in [104], where black hole solutions hovering outside a black brane horizon were found. In this latter case the tendency of the object to fall towards the horizon is balanced by the electrostatic force towards the boundary due to a charged defect in the 3d dual CFT.

In addition, one of the massless vectors in our theory, the Betti vector, comes from internally wrapped branes, and it is dual to a baryonic symmetry in the dual field theory [85]. None of the $U(1)$ gauge fields in the others models we mentioned are dual to baryonic symmetries. We will revisit these points later when we deal with the phase transitions in the canonical ensemble.

3.4 Black hole thermodynamics

The thermodynamic quantities associated to the black hole are

$$S = \pi r_H^2, \quad T = \frac{1}{4\pi} h'(0) e^{-\frac{\beta(0)}{2}}. \quad (3.42)$$

The entropy is given by the Bekenstein-Hawking formula (with $G_N = 1$), the temperature (in units where the Boltzmann constant is 1) can be derived from the periodicity of the Euclidean time coordinate $\tau = it$ for which the metric (3.24) is regular, and the mass can be inferred from the AdS Reissner-Nordström solution (3.38). Furthermore, the electromagnetic charges were defined in (3.27) and the corresponding electrostatic and magnetostatic potentials are (in a gauge for which the vector potentials vanish on the boundary of AdS)

$$\phi^{\mathcal{A}_i} \equiv - \int_{r_h}^{\infty} F_{\mathcal{A}_i, tr} dr = \mathcal{A}_{i,t}(r_h) = \xi_i(r_h), \quad \chi_{\mathcal{A}_i} \equiv - \int_{r_h}^{\infty} G_{\mathcal{A}_i, tr} dr. \quad (3.43)$$

The first law of thermodynamics relates these quantities along a family of black hole solutions, in this way:

$$dM = T dS + \phi^{\mathcal{A}_i} dQ_i, \quad (3.44)$$

where the mass of the black hole $M = -h_{(3)}/2 + \dots$ receives contributions from the light scalars, as we will see below. The relation (3.44) can be checked analytically for AdS Reissner-Nordström (3.38). For our numerical solutions, it provides a nontrivial consistency check. Indeed, the thermodynamic quantities can be computed from the behavior of the fields either close to the black hole

horizon or near the boundary of AdS. The relation (3.44) indicates that they are not unrelated: they are correlated by the existence of a solution to the equations of motion that interpolates between these distant regions and is regular everywhere.

3.4.1 Renormalized on-shell action

Our Lagrangian contains a Higgsed vector field and scalar fields which are dual to irrelevant operators. Therefore, we must take care to identify the correct counterterms and obtain a finite result for the on-shell action. Holographic renormalization in presence of massive vector fields was worked out in [105], where the necessary counterterms to renormalize the Proca-AdS action were obtained via the Hamilton-Jacobi formalism. Moreover, vector fields acquiring mass via spontaneous symmetry breaking were considered in [106, 107].

As explained in the previous sections, the diverging modes for all but the lightest scalar fields must be required to vanish in order not to spoil the AdS asymptotics¹⁰. In the Hamilton-Jacobi procedure for holographic renormalization [109] the vanishing of the diverging modes can be formulated as a set of second-class constraints, ensuring consistency (see for instance [110, 111]). This means that, when deriving the equations of motion using the variational principle, the coefficients of the non-normalizable modes will be fixed to zero.

Provided these constraints are satisfied, the counterterms that renormalize the action are the Gibbons-Hawking term I_{GH} , the canonical counterterms I_{ct} and the counterterm $I_{ct,A}$ due to the presence of the massive vector field as in [105]:

$$I_{ren} = I + I_{GH} + I_{ct} + I_{ct,A} . \quad (3.45)$$

The term I_{GH} is of the form

$$I_{GH} = \frac{1}{2} \int_{\partial M} d^3x \sqrt{g_3} \Theta , \quad \Theta_{\mu\nu} = -(\nabla_\mu n_\nu + \nabla_\nu n_\mu) . \quad (3.46)$$

where $g_{3,ab}$ is the induced metric on the boundary ∂M , Θ is the trace of the extrinsic curvature, and n_μ is the unit vector normal to the boundary. The term I_{ct} contains the counterterms necessary to cancel the divergences [112]

$$I_{ct} = \int_{\partial M} d^3x \sqrt{g_3} \left[\frac{l}{2} \mathcal{R} - \frac{l^3}{2} \left(\mathcal{R}_{bc} \mathcal{R}^{bc} - \frac{3\mathcal{R}^2}{8} \right) + \mathcal{W}(\phi) \right] , \quad (3.47)$$

¹⁰ One could however turn on the sources for these irrelevant operators perturbatively, as done for example in [108]. We thank A. Bzowski, Y. Korovin and I. Papadimitriou for discussions about this point.

where \mathcal{R}_{ab} denotes Ricci curvature on the boundary ∂M . The radius of AdS in our units is $l = 1/2$. The superpotential \mathcal{W} appearing in (3.47) satisfies this relation:

$$V = \frac{1}{2} \left(-\frac{3}{2} \mathcal{W}^2 + g^{ij} \partial_i \mathcal{W} \partial_j \mathcal{W} \right). \quad (3.48)$$

For our purposes, it is sufficient to know the form of \mathcal{W} close to the AdS vacuum. More precisely, we can write the scalar potential in terms of the fields ϕ_{m^2} which (1) have canonical kinetic terms in a neighborhood of the minimum of the potential, and (2) diagonalize the mass matrix. In terms of these fields, the superpotential is

$$\mathcal{W}_{\pm} = 4 + a_{\pm} \phi_{-2,a}^2 + b_{\pm} \phi_{-2,b}^2 + c_{\pm} \phi_{18}^2 + d_{\pm} \phi_{10}^2 + e_{\pm} \phi_4^2 + \mathcal{O}(\phi^3). \quad (3.49)$$

The modes with $m^2 l^2 = 4, 10, 18$ fall off faster than $1/\sqrt{g_3}$ near the boundary of AdS and hence do not contribute to (3.47). For the light modes with $m^2 l^2 = -2$, the coefficients a_{\pm} and b_{\pm} are the conformal dimensions of the operators dual to these scalar fields [109]. Each of them can be 1 or 2. The divergences in the action cancel if we use $a = b = 1$ (see for example the discussion in [113]). Therefore, it is sufficient to take into account

$$\mathcal{W} = \mathcal{W}_{-} = 4 + \phi_{-2,a}^2 + \phi_{-2,b}^2 + \dots \quad (3.50)$$

Finally, following the prescription of [106, 107, 105], the presence of the massive vector field requires the presence of an additional counterterm of the form

$$I_{ct,A} \propto \int_{\partial M} d^3x \sqrt{g_3} \mathcal{B}_{\mu} \mathcal{B}^{\mu}. \quad (3.51)$$

However, in our subspace of solutions this counterterm does not give any finite contribution to the renormalized action, as one can see from the asymptotic expansion of ζ in (3.40).

3.4.2 Electric solution

We now compute the on-shell value of the renormalized action for purely electric black holes, following [114, 115]. The magnetic ones follow along the same lines.

For purely electric and purely magnetic configurations, the terms of the form $F \wedge F$ in the action (3.1) vanish. Substituting the trace of the Einstein

equation into the action (3.1), we get

$$\begin{aligned} I &= \int d^4x \sqrt{g} \left(\frac{1}{2} \text{Im} \mathcal{N}_{\Lambda\Sigma} F_{\mu\nu}^\Lambda F^{\mu\nu\Sigma} + V \right) \\ &= \int d^4x \sqrt{g} \left(R_t^t + 2 \text{Im} \mathcal{N}_{\Lambda\Sigma} F_{tr}^\Lambda F^{tr\Sigma} - 2 A_t^\Lambda A^{t,\Sigma} k_\Lambda^u k_{\Sigma u} \right). \end{aligned} \quad (3.52)$$

In the last equation, we have used the tt component of the Einstein equations. Remarkably, the quantity R_t^t can be written as a total derivative [115]

$$R_t^t = \frac{1}{\sqrt{g}} \frac{\partial}{\partial r} (\sqrt{g_3} \Theta_t^t). \quad (3.53)$$

Moreover, we make use of Maxwell's equations for the vectors. Eq. (3.52) assumes the following form

$$\begin{aligned} I &= 8\pi T \int dr \left[\sqrt{g_3} \frac{\Theta_t^t}{2} + G_{\mathcal{A}_i} \xi_i + G_\zeta \zeta \right]' \\ &= 8\pi T \left(\sqrt{g_3} \frac{\Theta_t^t}{2} + \mathcal{A}_{t,i} Q_i \right) \Big|_{r_H}^{r_C}, \end{aligned} \quad (3.54)$$

where we have regulated the action using a radial cutoff r_C which will be sent to infinity after the integration. Notice that the expression (3.54) gives contributions both at the horizon, located at $r = r_H$ and at the boundary.

Adding the counterterms to this action, we have

$$I_{GH} = (4\pi T) \frac{1}{2} r_C e^{-\beta(r_C)/2} [r h' + h(4 - r\beta')]\Big|_{r_C}. \quad (3.55)$$

Moreover, using $\mathcal{R} = 2/r^2$ and $\mathcal{R}_{ab}\mathcal{R}^{ab} = 2/r^4$, the counterterms action (3.47) becomes

$$I_{ct} = \frac{1}{2} (4\pi T) e^{-\beta/2} \sqrt{h} [1 + 2r^2(4 + \phi_{-2,a}^2 + \phi_{-2,b}^2 + \dots)] \Big|_{r_C}. \quad (3.56)$$

The complete renormalized on-shell action I_{ren} (3.45) can thus be calculated using the asymptotic and horizon expansions of the fields (3.40) and (3.41). We obtain

$$\frac{I_{ren}}{4\pi T} = -\frac{1}{2} (h_{(3)} - 4e_{a,2} e_{a,1} - 4e_{b,2} e_{b,1}) - \frac{1}{4} [r^2 e^{-\beta/2} h']_{r_H} - \phi^{Ai} Q_i, \quad (3.57)$$

where $e_{a,1}, e_{b,1}$ are the leading fall-off at the boundary of the light scalar modes ($m^2 l^2 = -2$)¹¹

$$\phi_{-2,a} = \frac{e_{a,1}}{r} + \frac{e_{a,2}}{r^2} + \mathcal{O}(r^{-3}) , \quad \phi_{-2,b} = \frac{e_{b,1}}{r} + \frac{e_{b,2}}{r^2} + \mathcal{O}(r^{-3}) . \quad (3.59)$$

The right-hand side of (3.57) can be interpreted as the expression for the free energy, as anticipated. The term evaluated at r_H is nothing but TS , using (3.42). Furthermore, we can identify the mass of the black hole

$$M = -\frac{h_{(3)}}{2} + 2(e_{a,2}e_{a,1} + e_{b,2}e_{b,1}) . \quad (3.60)$$

Therefore, we can rewrite (3.57) as

$$\frac{I_{ren}}{4\pi T} = M - TS - \phi^{A_i} Q_i . \quad (3.61)$$

This expression for the mass agrees with the one obtained from the renormalized boundary stress-energy tensor τ^{ab} [116]

$$M = Q_t = \frac{1}{16\pi} \int_{\Sigma} \sqrt{\sigma} u_a \tau^{ab} \xi_t , \quad \tau^{ab} = \frac{2}{\sqrt{g_3}} \frac{\delta I}{\delta g_{3,ab}} , \quad (3.62)$$

where $\xi^a \partial_a = \partial_t$ is the Killing vector of the time translation isometry of the boundary metric $g_{3,ab}$, Σ is a constant time slice on the boundary ∂M with induced metric σ , and u^a is the timelike unit normal vector to Σ on ∂M (see for instance [117]). We have:

$$\tau^{tt} = -(\Theta^{ab} - \Theta g_3^{ab}) + \mathcal{W} g_3^{ab} - l \left(\mathcal{R}^{ab} - \frac{1}{2} g_3^{ab} \mathcal{R} \right) \quad (3.63)$$

which yields exactly (3.60).

We have seen then that the choice of counterterms (3.46)-(3.47) reproduces the Gibbs free energy (3.61). From [114] one can see that I_{ren} is stationary for fixed temperature and chemical potential, and in particular for fixed $e_{a,1}$ and $e_{b,1}$, as is the case in our solutions¹².

¹¹ The values of the mass eigenstate coefficients $e_{a,i}$, $e_{b,i}$ are related to the expansion parameters we used in (3.40b) as follows:

$$e_{a,1} = -\sqrt{3} b_{(1)} , \quad e_{a,2} = -\sqrt{3} b_{(2)} , \quad (3.58)$$

$$e_{b,1} = -\sqrt{3} \tau_{(1)} , \quad e_{b,2} = -\frac{\sqrt{3}(3b_{(1)}^2 + 2(\tau_{(1)}^2 + 5\tau_{(2)})}{10} .$$

¹² It is nevertheless possible to impose different boundary conditions (Neumann, mixed) for the scalar fields with $m^2 = -2$, by choosing appropriate boundary counterterms. For simplicity we restrict here to the case of fixed $e_{a,1} = e_{b,1} = 0$ fixed, but it would be interesting to analyze the other cases as well.

Finally, let us mention that if we are instead interested in the canonical ensemble, we need to add to the action I_{ren} obtained before the additional finite counterterm

$$I_{HR} = - \int d^3x \sqrt{g_3} \text{Im} \mathcal{N}_{\Lambda\Sigma} n_\mu F^{\mu\nu, \Lambda} A_\nu^\Sigma, \quad (3.64)$$

called Hawking-Ross counterterm [118]. The total action $I_{ren} + I_{HR}$ is then stationary for fixed electric *charges*, hence the first law (3.44) with Helmholtz free energy

$$F_{Helmholtz} = M - TS. \quad (3.65)$$

The on shell action for the purely magnetic configuration can be worked out analogously, for instance along the lines of [119, 120].

We have tested the accuracy of our numerics by verifying that the first law (3.44) is satisfied for infinitesimal changes of the conserved quantities δM , δS , δQ_i . We have moreover computed the renormalized on shell action by numerically integrating (3.54) and found agreement with the expression we obtained in eq.(3.61).

3.5 Canonical ensemble

In what follows we analyze the thermodynamics of the novel black hole solutions in the canonical ensemble, namely for fixed values of temperature and electromagnetic charges. Moreover, in what follows, we restrict to configurations with boundary conditions $e_{a,1} = e_{b,1} = 0$ for the scalar fields of mass $m^2 l^2 = -2$. We consider the thermodynamic analysis of the purely electric configuration, since (as opposed to the purely magnetic ones) these have a nontrivial profile for the massive vector field. For simplicity, we will moreover restrict in our discussion to solutions with $Q_1 = 0$: allowing for a nonzero value of the other charge Q_2 is sufficient for the solution to support a nontrivial massive vector profile¹³

We compare configurations with fixed electric charge Q_2 , but with different values of the radius of the event horizon (hence different values for the entropy and mass of the black hole). Whenever there are multiple solutions with the same T and Q_2 , those which minimize the Helmholtz free energy (3.65) will dominate the thermodynamic ensemble.

¹³ Setting $P^2 = Q_2 = 0$ (and keeping $e_{a,1} = e_{b,1} = 0$) only yields the AdS-Reissner-Nordström solution.

We were able to find families of black holes by sampling the space of solutions with different discrete values of the event horizon. The results of this procedure are plots like those in Figure 3.4. It turns out that for values $|Q_2| \geq Q_c$ where $Q_c \approx 0.17$, the temperature plotted in function of the black hole entropy is a monotonically increasing function, while if we lower the charge to values $|Q_2| < Q_c$, for a suitable temperature range we are able to find three branches of solutions, characterized by three possible different values of the entropy. We call them small, medium and large black holes, where the size relates to the black hole radius as compared with the AdS radius. In our conventions, this happens for black hole entropies of order 1, see Figure 3.4.

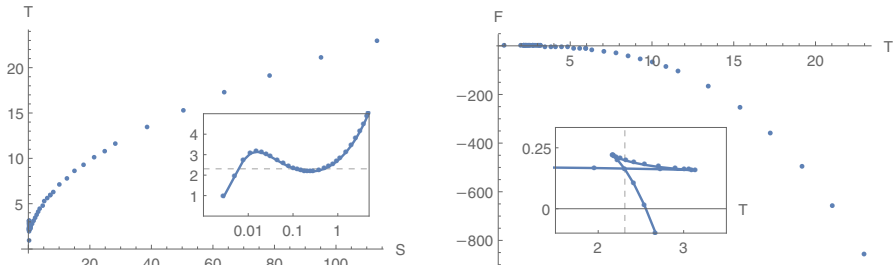


Figure 3.4: Plot of the temperature in function of the black hole entropy for the set of solutions with $Q_2 = 0.15$. At the temperature $T \approx 2.4$ the derivative of the free energy exhibits a discontinuity, revealing a first order phase transition. Notice that the horizontal axis in the inset of the first plot is logarithmic, hence the Maxwell area law is not explicitly visible.

With reference to the same figure, in the right panel we plot the free energy as a function of the temperature, for the same set of solutions. It is clear that for $Q_2 < Q_c$ the free energy exhibits a discontinuity in the first derivative for a value of temperature $T \approx 2.4$ (see Figure 3.4). This signals the onset of a small-large black hole first order phase transition, in all similarity with the phase transition for Reissner-Nordström in AdS spacetime found in the seminal papers of [121, 122] and [117]. The phase transitions become a crossover for charges $Q_2 > Q_c$, while second order for the critical charge $Q_2 \approx 0.17$. Notice that, despite appearing almost horizontal in the plot, the free energy for each branch is always monotonically decreasing, as it should be since $\partial F / \partial T = -S$. Lastly, the medium-sized black holes are always thermodynamically disfavored since their free energy is always greater than that of the other two black hole branches. They also have negative specific heat

$$C_S = T \left(\frac{\partial S}{\partial T} \right)_{Q,T} \quad (3.66)$$

while the small and large have positive specific heat.

It is instructive to plot the behavior of the massive vector field for the three different black hole branches, as done in Figure 3.5. We notice that, with reference to the black hole family in Figure 3.4, the small black holes (blue line in the plot) have a profile for the massive vector with two extrema: one at a positive value close to the black hole horizon and a smaller peak at a negative value somewhat further away. The first peak goes away as the size of the black hole increases. The medium black holes (orange in Figure 3.5) have only a minimum in the ζ -profile, which is however more pronounced and closer to the black hole horizon (in terms of u , i.e. with r_H scaled out). For large black holes (in green), the minimum in the massive vector profile becomes ever less pronounced. It settles at $u_{\text{extr}} \approx 0.34$, which corresponds to $r_{\text{extr}} \approx 1.4r_H$. To sum up, during the phase transition from small to large black holes, the massive vector field decreases in absolute value and the radial coordinate r_{extr} corresponding to its maximum value increases. Hence it gets “expelled further” from the black hole horizon.

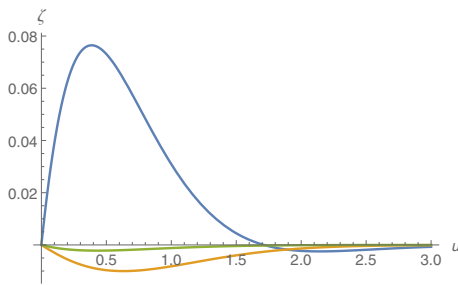


Figure 3.5: Plot of the radial profile for the massive vector field for small black holes (blue, $r_H = 0.03$), medium ones (orange, $r_H = 0.07$) and large ones (green, $r_H = 0.14$).

We now analyze the scalar field asymptotic expansion. The mode $e_{b,1}$ corresponds to the expectation value of an operator of conformal dimension 2, $\langle \mathcal{O}_2 \rangle = -\tau_{(2)}/2$. This is the order parameter of our phase transition. The value of $\tau_{(2)}$ in function of the temperature is visualized in Figure 3.6. We see that its absolute value decreases during the phase transition from low to high temperature. Moreover, its behavior resembles that of the isotherms for the Van der Waals system (liquid/gas -like phase transition). This is reminiscent of what happens for black holes solutions of Fayet-Iliopoulos gauged supergravity [123, 124].

The interpretation of the value of ζ_4 is more subtle: due to interaction terms with the light scalar fields, the term proportional to ζ_4 does not dominate

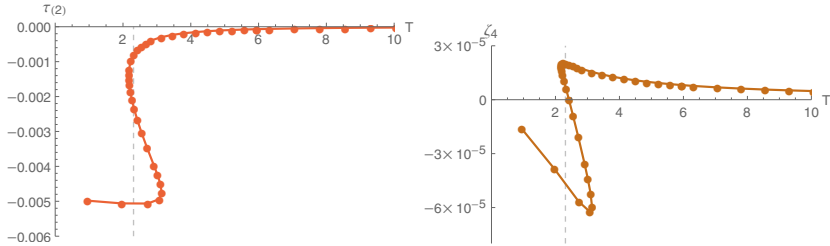


Figure 3.6: On the left, we plot of the expectation value of the operator of dimension 2 dual to the light scalar mode with mass $m^2 l^2 = -2$ in function of the temperature of the black hole. On the right the plot of the parameter ζ_4 in function of the temperature.

its asymptotic expansion (3.40c). Its interpretation as the expectation value of the corresponding operator with $\Delta = 5$ needs verification, by means of the identification of the correct renormalized conjugate momenta of ζ , see for instance [106, 125]. We nevertheless provide the behavior of ζ_4 in the second graph of Figure 3.6, where we can see once again that the small-large black hole phase transition manifest itself as a decrease in the absolute value of this parameter.

We conclude by highlighting yet another difference with respect to the holographic superconductor phase transition. The process we have described here for the new class of solutions involves two phases where the condensate is never vanishing, namely the massive vector field is always switched on. There is no restoring of the broken symmetry for a finite temperature, as opposed to [23, 24, 25, 103], where the preferred phase for high temperatures is the scalarless Reissner-Nordström solution, with no scalar condensate.

3.6 Conclusion and outlook

In this work we have constructed novel numerical solutions of $\mathcal{N} = 2$ gauged supergravity coupled to vector and hypermultiplets. This four-dimensional theory arise as consistent truncation of M-theory on the manifold M^{111} and it is endowed with one Betti vector multiplet. The presence of the latter, corresponding to light degrees of freedom (two scalars of mass $m^2 = -2$ and one massless vector), allows for the construction of black hole solutions with non vanishing massive vectors. This fact was noticed in the BPS case as well [86] and it would be interesting to understand its deeper origin, in relation to brane world volume gauge theories.

We have moreover analyzed the thermodynamics of the black hole configurations, revealing two branches of stable solutions: the so-called small black holes and the large ones. A small-large black hole phase transition was found, during which the massive vector field decreases in absolute.

The black holes constructed here serve as the starting point for future analysis of bound states in AdS spacetimes, in view of applications to glassy systems [26]. The next step in this direction will be to establish the possible existence of finite temperature bound states composed of the black holes backgrounds which we have discovered, surrounded by smaller probe black holes. In our case, additional interactions between the probes and the massive vector field condensate are present, and we expect this new feature to play a role in the equilibrium condition for the charged probes. It would moreover be interesting to quantify the effect of strings stretched from the horizon to the probes, and among the probes themselves. Such solutions which manifest themselves in the lower dimensional supergravity system as vortex-like solutions, like those constructed in [126] in the context of AdS black holes.

Once these points are addressed, one can then study and map the parameter space of allowed stable and metastable configurations. Subsequently, one can extract the relaxation dynamics of such bound systems and verify if they exhibit logarithmic aging behavior which is typical of many amorphous systems. The overall picture emerging from [26] was that a liquid, single-centered horizon corresponding to the liquid phase of matter, can turn into a fragmented, disordered one corresponding to a glassy phase. It would be interesting to compute holographic transport coefficients for the composite systems, such as shear viscosity and conductivity, to confirm this picture. We hope to report back on all these points in the near future.

Chapter 4

De Sitter holography

Our Universe has a positive cosmological constant [16, 17] which is currently causing it to accelerate its expansion. If general relativity remains a good approximation at cosmological scales, all sources of gravitational attraction will be diluted away and the expansion will keep on accelerating. In the end, our Universe will then be left mostly empty, ever closer to a de Sitter space.

The de Sitter solution to Einstein gravity is very similar to its [AdS](#) cousin. It has a conformal boundary and a corresponding asymptotic scaling symmetry. In AdS holography, these properties are essential for its relation to CFTs and for the formulation of the Witten dictionary. We will see in this chapter that an analogous duality has been proposed for de Sitter space. The motivation for studying these proposals is obvious: where several other attempts have failed or remained inconclusive, these proposals potentially provides a class of tractable [UV](#)-complete models of quantum gravity with positive cosmological constant.

In this chapter we will introduce de Sitter space and discuss quantum field theory on a fixed de Sitter background. We then consider some tools that are used to incorporate gravity in the quantum theory. We will review a proposal for a wave function that describes the state of universes with dynamical gravity. This will allow us to formulate the [dS/CFT](#) proposal, both in the language of correlation functions and in terms of the wave function. To conclude this chapter, we compare this proposal with AdS/CFT and find some important differences. These observations set the course for the remainder of this thesis.

4.1 De Sitter space (dS)

De Sitter space is the maximally symmetric solution to Einstein's equations with a positive cosmological constant. With d spatial and one time dimension, dS_{d+1} satisfies (2.2) but with $l_{\text{AdS}} \rightarrow il_{\text{dS}}$. It contains the same number of isometries. Like AdS_{d+1} , it can be embedded in $d + 2$ -dimensional flat space. In this case, that higher-dimensional space is Minkowski $_{d+2}$ with embedding¹

$$\eta_{MN} X^M X^N = l_{\text{dS}}^2, \quad \eta_{MN} = \text{diag}(-1, 1, \dots, 1). \quad (4.1)$$

This is depicted in Figure 4.1. The isometries of de Sitter are the isometries of Minkowski space which preserve this embedding surface,

$$L_{MN} = X_M \partial_N - X_N \partial_M, \quad [L_{MN}, L_{PQ}] = \eta_{NP} L_{MQ} + \dots, \quad (4.2)$$

which generate the group $SO(1, d + 1)$.

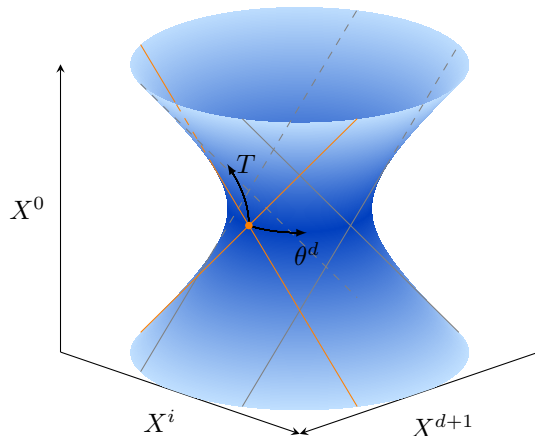


Figure 4.1: Slide of the embedding of dS_{d+1} in $d + 2$ -dimensional Minkowski space as the hypersurface (4.1). The light cone emanating from the origin of the global coordinate system – indicated here as T and θ^d – is drawn in orange. The causal horizons of an observer on the north pole $\theta^d = 0$ are indicated in dashed gray lines. The isometries of dS are the rotations and boosts in the embedding space that leave the origin invariant.

The full space can be covered with *global coordinates* (T, θ^i) , where time T ranges from $-\pi/2$ in the infinite past to $\pi/2$ in the far future. These coordinates

¹ For $d = 1$, this is exactly the same as (2.3)-(2.4) with the same l . The minus sign only indicates which direction is to be interpreted as time. Geometrically the compact version of AdS_{1+1} is identical to dS_{1+1} .

are related to the embedding space coordinates through

$$X^0 = l \tan T, \quad X^I = l \frac{\omega^I}{\cos T}, \quad (4.3)$$

where the $d+1$ coordinates ω^I are given in terms of angles θ^i as in (2.6). The metric on dS_{d+1} induced by this embedding is

$$\text{d}s^2 = l^2 \frac{-\text{d}T^2 + \text{d}\Omega_d^2}{\cos^2 T}, \quad -\frac{\pi}{2} \leq T \leq \frac{\pi}{2}. \quad (4.4)$$

Note that the proper time along timelike curves can still be infinite in this metric, even though T only has finite range. The trajectories of light rays are easily understood in these coordinates, and the situation is radically different from that in AdS. There, null trajectories can span infinite spatial distance in finite global time. In dS, light rays cannot travel more than halfway around the spatial S^d even if they are given all the time in the Sitter Universe. The situation is depicted in the Penrose diagram in Figure 4.2, which conformally represents de Sitter in the coordinates (4.4).

Since not any two points in a de Sitter spacetime can communicate (assuming as usual that the speed of light is the Universal speed limit on information exchange) one needs to know the trajectory of observers in global dS to determine which information they can access. For each observer, this causal information is summarized by two “points” on the Penrose diagram: its intersection with the asymptotic past \mathcal{I}^- and with the infinite future \mathcal{I}^+ . The former dictates which events can be influenced by the observer, while the latter determines which events can be observed. The intersection of these regions is called the *causal diamond*, whereas the boundaries are *causal horizons*. Because of their cosmological origin, and to distinguish them from black hole horizons, these are called *cosmic horizons*. By definition they are observer dependent and non-local, in the sense that they can only be defined once the full history of the universe is known.

On the Penrose diagram in Figure 4.2, we have indicated the causal diamond of an observer whose world line starts and ends on the north pole of S^d , given by $\theta^d = 0$. This part of de Sitter space can be described by coordinates (t, r, θ^a) with

$$X^0 = \sqrt{l^2 - r^2} \sinh \frac{t}{l}, \quad X^i = r \omega^i, \quad X^{d+1} = \sqrt{l^2 - r^2} \cosh \frac{t}{l}. \quad (4.5)$$

The time coordinate t ranges over the full real line, but r is restricted between 0 and l_{dS} . The metric in these coordinates is the analytic continuation $l_{\text{AdS}} \rightarrow -il_{\text{dS}}$ of the AdS metric in global coordinates (2.8),

$$\text{d}s^2 = -\left(1 - \frac{r^2}{l^2}\right) \text{d}t^2 + \left(1 - \frac{r^2}{l^2}\right)^{-1} \text{d}r^2 + r^2 \text{d}\Omega_{d-1}^2. \quad (4.6)$$

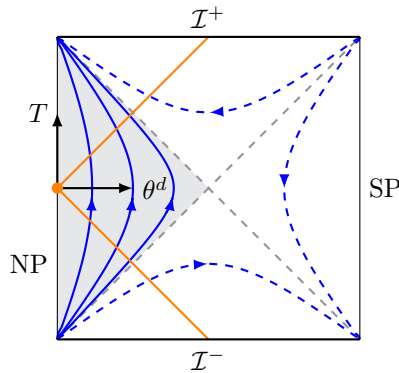


Figure 4.2: Penrose diagram of de Sitter space containing the same indications as Figure 4.1. The horizontal axis is the θ^d direction, whereas the vertical axis corresponds to T . The infinite future \mathcal{I}^+ and past \mathcal{I}^- are at finite distance in this conformal diagram. It is important to note that these horizontal lines are of a completely different nature than the vertical lines that make up the outer square of this diagram. Indeed, the latter correspond to the north and south pole (NP and SP) of the spatial sphere. They are completely regular points of the spacetime. The loci \mathcal{I}^\pm , on the other hand, are asymptotically far in the past and future. The causal diamond of an observer who reaches \mathcal{I}^\pm on the north pole, corresponds to the shaded gray region. This is the patch on which the coordinates (4.5) are valid. The continuous blue lines correspond to static patch time evolution at fixed r . The dashed blue lines are the continuation of this isometry in global de Sitter.

Time translations are symmetries of this line element. The patch covered by these coordinates is therefore called the *static patch*.

Static patch time translations are generated by $L_{0,d+1}$ in (4.2), which traces out the blue curves in Figure 4.2. Its direction is forward in global time T near the north pole, towards the global past near the south pole, and spacelike in other regions of dS. This behavior extends to other linear combinations of the isometries (4.2): there is no globally timelike dS isometry. Correspondingly, if a field excitation near the north pole contributes positively to the conserved charge associated to $L_{0,d+1}$, the same field excitation rotated to south pole contributes negatively. On global de Sitter space, there is no positive conserved energy [19].

There are many other coordinates systems that find their use to describe de Sitter space (see for example [127]) but we will mention just one more. Consider

the coordinates $\eta < 0$ and $x^i \in \mathbb{R}^d$,

$$X^0 = \frac{l^2 - \eta^2 + x^2}{-2\eta}, \quad X^i = l \frac{x^i}{-\eta}, \quad X^{d+1} = \frac{l^2 + \eta^2 - x^2}{-2\eta}, \quad (4.7)$$

where $x^2 = \delta_{ij}x^ix^j$. The metric in terms of these coordinates is

$$ds^2 = l^2 \frac{-d\eta^2 + \delta_{ij}dx^idx^j}{\eta^2} = -d\tau^2 + e^{2\tau/l} \delta_{ij}dx^idx^j, \quad (4.8)$$

where $\tau = -l \log(-\eta/l)$ is the proper time of an observer on the north pole $x^i = 0$. The factor $e^{2\tau/l}$ multiplying the spacial coordinates is called the *scale factor*, and is often denoted as $a(\tau)$. Since spatial slices are infinite flat planes, they are called the *planar slicing* of the part of dS indicated in gray on Figure 4.3. More precisely, this is called the *northern future planar patch*, which covers the region where $X^0 > X^{d+1}$, or equivalently $T > \theta^d - \frac{\pi}{2}$. These coordinates are related to the global coordinates as follows,

$$\eta = \frac{-\cos T}{\sin T + \cos \theta^d}, \quad |x| = \frac{\sin \theta^d}{\sin T + \cos \theta^d}. \quad (4.9)$$

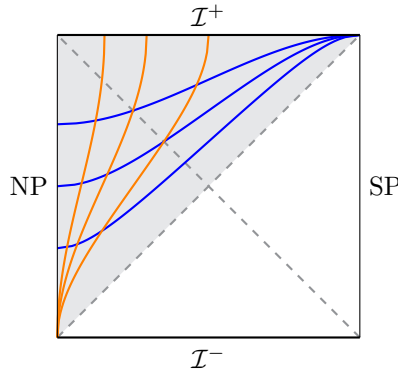


Figure 4.3: Penrose diagram of de Sitter space with the patch covered by the planar coordinates (4.7) shaded gray. The curved blue lines are slices of constant η , ranging from $\eta \rightarrow -\infty$ at the causal horizon to $\eta \rightarrow 0$ at \mathcal{I}^+ . The orange lines are trajectories of observers at constant x .

There is an elegant generalization of the Pythagorean theorem for the geodesic distance d between two points in de Sitter space [127]. It reduces to each of the line elements we have given before, whenever the two points are infinitesimally close to each other. First, consider the embedding (2.6) of a hypersphere in Euclidean space. The inner product $\delta_{ij}\omega_1^i\omega_2^j$ is given by the

cosine of the relative angle θ_{12} between the two unit vectors. This angle is exactly the geodesic distance on the unit sphere between the two points labeled by ω_1 and ω_2 . In terms of the coordinates θ^a , we have the recursive relation

$$\omega_1^d \cdot \omega_2^d = \cos \theta_1^d \cos \theta_2^d + \sin \theta_1^d \sin \theta_2^d \omega_1^{d-1} \cdot \omega_2^{d-1}. \quad (4.10)$$

This argument can be applied to de Sitter space as well. In terms of the quantity $P \equiv \eta_{MN} X_1^M X_2^N / l^2$, the geodesic distance d is given as $P = \cos(d/l)$. We will often just refer to the quantity P as the invariant distance between two points in dS. In global coordinates, it is given by

$$P(T_1, \theta_1; T_2, \theta_2) = \frac{\omega_1 \cdot \omega_2 - \sin T_1 \sin T_2}{\cos T_1 \cos T_2}, \quad (4.11)$$

where ω^I are the embedding coordinates (2.6) of the hypersphere. In other coordinate systems, there exist equivalent expressions which are valid only within the patch covered by those coordinates. For example in the planar patch,

$$P(\eta, \vec{x}; \eta', \vec{y}) = \frac{\eta^2 + \eta'^2 - (\vec{x} - \vec{y})^2}{2\eta\eta'}. \quad (4.12)$$

The geodesic distance d , and by extension P , encodes a lot of the causal structure of the spacetime. The spatial part (4.10) of P is always between -1 and 1 , which makes d at $T_1 = 0 = T_2$ real and positive. Whenever the two points are timelike separated, P is larger than 1 , making d^2 negative. Furthermore, if the two points are antipodally timelike separated – this means one of the points is timelike separated from the spacetime antipode of the other – P is smaller than -1 . This again makes d^2 negative.

As a final note, de Sitter space can be conformally completed by the addition of a conformal boundary, as was possible in AdS. In the case of global dS, this conformal boundary consists two disconnected pieces: the asymptotic future \mathcal{I}^+ and past \mathcal{I}^- . When restricting to the planar patch, only one of those conformal boundaries is included in the conformal compactification. It is located at $\eta = 0$.

4.2 A scalar field on the planar patch

We will now consider scalar field propagating on a fixed de Sitter background. Using this simple example, we will develop the necessary tools and insight to formulate the dS/CFT correspondence. We will use units in which $l_{\text{dS}} = 1$.

4.2.1 Free field theory

The action of a free scalar field with mass m^2 on the planar patch (4.8) of $d + 1$ -dimensional de Sitter space in units with $l_{\text{dS}} = 1$ is

$$S = -\frac{1}{2} \int d\eta d^d x \frac{1}{(-\eta)^{d+1}} [-\eta^2 (\partial_\eta \phi)^2 + \eta^2 (\nabla \phi)^2 + m^2 \phi^2] . \quad (4.13)$$

A general solution to the linear (free) equations of motion can be written as a sum of mode functions $\phi_\lambda(\eta, x)$, where λ is a label to be specified later. For the action (4.13), the mode functions can be obtained by first diagonalizing the spatial Laplacian and then solving the equations of motion for the η -dependent coefficients. In other words, we can decompose the field in spatial Fourier modes,

$$\phi_\lambda(\eta, x) = \phi_{(\vec{\lambda}, \vec{k})}(\eta, x) = \varphi_{(\vec{\lambda}, k)}(\eta) \frac{e^{i\vec{k} \cdot \vec{x}}}{(2\pi)^{\frac{d}{2}}} , \quad (4.14)$$

with time-dependent coefficients. For this free theory, the action becomes an integral over \vec{k} and the resulting equations of motion for each of the modes decouple indeed,

$$(-\eta)^{d+1} \partial_\eta \left((-\eta)^{1-d} \partial_\eta \varphi_{(\vec{\lambda}, k)} \right) + (\eta^2 k^2 + m^2) \varphi_{(\vec{\lambda}, k)} = 0 . \quad (4.15)$$

This is an equation of the Bessel type, which has two linearly independent solutions, which can be expressed in terms of several types of Bessel functions. To start, we will express the general solution as a combination of Hankel functions of the first and second kind,

$$\varphi_k = \frac{\sqrt{\pi}}{2} (-\eta)^{\frac{d}{2}} H_\nu^{(1)}(-k\eta) , \quad \bar{\varphi}_k = \frac{\sqrt{\pi}}{2} (-\eta)^{\frac{d}{2}} H_\nu^{(2)}(-k\eta) , \quad (4.16)$$

where $\nu \equiv \sqrt{d^2 - 4m^2}/2$. Since $-k\eta > 0$, these modes are each others complex conjugate.

To get a sense of how the modes (4.16) behave, consider that they have fixed coordinate wavelength k , but due to the constant expansion in the planar patch – the factor $1/\eta^2$ in the metric – the physical wavelength grows. Therefore, for each of these modes there was a time at which their physical wavelength was well shorter than the dS length l_{dS} . In that regime, their propagation is insensitive to the curvature of spacetime. Indeed, in the limit $-k\eta \rightarrow \infty$ the modes (4.16) oscillate, as they would on flat space,

$$\varphi_k \propto \frac{(-\eta)^{\frac{d}{2}}}{\sqrt{-k\eta}} e^{-ik\eta} , \quad \bar{\varphi}_k \propto \frac{(-\eta)^{\frac{d}{2}}}{\sqrt{-k\eta}} e^{ik\eta} . \quad (4.17)$$

These modes decay in absolute value, which is related to the ongoing expansion of space. The energy with respect to τ per unit of comoving volume² is dominated at early times by the kinetic terms in (4.13), which are proportional to $k^2 e^{-2\tau} \varphi^2 \propto k^2 e^{-(d+1)\tau}$. This result is consistent with the energy of a photon gas in an expanding universe with scale factor a : it contains a factor a^{-d} related to the expanding volume of space, as well as a factor $1/a$ corresponding to the stretching of physical wavelength.

In the $-k\eta \rightarrow \infty$ limit, the mode φ_k oscillates like the positive frequency mode in Minkowski space, whereas $\bar{\varphi}_k$ behaves like the negative frequency mode. The precise way to distinguish positive from negative frequency modes is to calculate the sign of the Klein-Gordon inner product of the modes (4.16). This is conveniently expressed in terms of the field's conjugate momentum $\pi(x)$,

$$(\phi_{\lambda_1}, \phi_{\lambda_2})_{\text{KG}} \equiv i \int d^d x (\phi_{\lambda_1}^* \pi_{\lambda_2} - \pi_{\lambda_1}^* \phi_{\lambda_2}) ,$$

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\eta \phi)} = (-\eta)^{1-d} \partial_\eta \phi(x) . \quad (4.18)$$

In the last line we used the specific form of the action (4.13). This inner product is preserved under time evolution and reduces to the Klein-Gordon (KG) inner product on flat space. The KG inner products of the modes (4.14) are

$$(\phi_{\vec{k}}, \phi_{\vec{k}'})_{\text{KG}} = \delta^{(d)}(\vec{k} - \vec{k}') , \quad (\bar{\phi}_{\vec{k}}, \bar{\phi}_{\vec{k}'})_{\text{KG}} = -\delta^{(d)}(\vec{k} - \vec{k}') , \quad (4.19)$$

whereas the KG inner product of ϕ with $\bar{\phi}$ vanishes. Therefore, φ_k are the positive frequency modes in this planar patch and $\bar{\varphi}_k$ are the negative frequency modes.

The late-time behavior $\eta \rightarrow 0$ is of interest as well. Whenever ν is not integer,³ the Hankel functions in (4.16) have modes two modes $\sim \eta^{\Delta_\pm}$ with $\Delta_\pm \equiv \frac{d}{2} \pm \nu$. We can isolate each of these components by taking linear

² This notion of energy does not correspond to a conserved charge, since translations of τ are not an isometry of the background. Furthermore, an infinitesimal unit of comoving, physical volume is proportional to $e^{d\tau}$ times the coordinate volume $d^d x$.

³ For integer ν , there can be logarithmic terms in the late-time expansion. An alternative way to obtain real mode functions is in terms of the Bessel-Y function, as $\varphi_+ - \varphi_-$. This remains finite for integer ν , but it generically contains not only the leading, but also a term proportional to the subleading fall-off.

combinations

$$\begin{aligned}\varphi_{(\Delta_+,k)} &\equiv \frac{k^{-\nu}}{\sqrt{2}}(\varphi_k + \bar{\varphi}_k) = \sqrt{\frac{\pi}{2}} k^{-\nu} (-\eta)^{\frac{d}{2}} J_\nu(-k\eta) , \\ \varphi_{(\Delta_-,k)} &\equiv \frac{k^\nu}{\sqrt{2}}(e^{i\pi\nu}\varphi_k + e^{-i\pi\nu}\bar{\varphi}_k) = \sqrt{\frac{\pi}{2}} k^\nu (-\eta)^{\frac{d}{2}} J_{-\nu}(-k\eta) .\end{aligned}\quad (4.20)$$

The KG inner products of these linear combinations are given by

$$\left(\phi_{(\Delta_+, \vec{k})}, \phi_{(\Delta_-, \vec{k}')} \right)_{\text{KG}} = i \sin(\pi\nu) \delta^{(d)}(\vec{k} - \vec{k}') , \quad (4.21)$$

whereas the other inner products of $\varphi_{(\Delta_+,k)}$ with itself and $\varphi_{(\Delta_-,k)}$ with itself vanish. This basis of solutions makes it particularly clear that the behavior of scalar field in de Sitter is analogous to that of scalar fields on AdS with Δ_\pm analytically continued $l_{\text{AdS}} \rightarrow il_{\text{dS}}$ and $-\eta$ playing the role of z .

4.2.2 Canonical quantization

The procedure of canonical quantization can be extended from flat space to the planar patch. We can again appeal to (4.17) and the argument that the modes (4.16) behave like the flat space modes $e^{\pm i\omega t}$. The procedure of canonical quantization then amounts to writing the a general field operator as a sum over positive and negative frequency mode functions, and promoting the coefficients to creation and annihilation operators,

$$\hat{\phi}(\eta, \vec{x}) = \int \frac{d^d k}{(2\pi)^{\frac{d}{2}}} \hat{\phi}_{\vec{k}}(\eta) e^{i\vec{k} \cdot \vec{x}} , \quad \hat{\phi}_{\vec{k}}(\eta) = \hat{a}_{\vec{k}} \varphi_k(\eta) + \hat{a}_{-\vec{k}}^\dagger \bar{\varphi}_k(\eta) . \quad (4.22)$$

The operators are declared to act on a Hilbert space of states and to satisfy the commutation relations of the Heisenberg algebra,

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}^\dagger] = \delta^{(d)}(\vec{k} - \vec{k}') . \quad (4.23)$$

There is a state $|E\rangle$ which is annihilated by all the $\hat{a}_{\vec{k}}$. It goes by many names, such as the *Euclidean vacuum* and the *Bunch-Davies vacuum* [128, 129, 130, 131, 132, 133, 134, 135, 136]. It merits a name because the identification of a “vacuum” state in dS is more subtle than in flat space. For example, it is not the minimum of a conserved energy, because there is no positive conserved energy in de Sitter space. The Euclidean vacuum is indeed the lowest eigenstate of the planar patch Hamiltonian H_η which generates translations in η , but this is not conserved. By a related argument, the Euclidean vacuum cannot be

characterized as “the state without any particles in it” either. Since different inertial observers accelerate with respect to each other in de Sitter space, they will disagree on what that “state without particles” is, because of the Unruh effect [137, 138, 139].

It is nevertheless possible to distinguish the Euclidean vacuum from other states by calculating the scalar field two-point function. Given the field operators (4.22), Wightman function in the Euclidean vacuum in momentum space is given by

$$\langle E | \hat{\phi}_{\vec{k}_1}(\eta) \hat{\phi}_{\vec{k}_2}(\eta') | E \rangle = \bar{\varphi}_{k_1}(\eta) \varphi_{k_2}(\eta') \langle E | a_{\vec{k}_1} a_{-\vec{k}_2}^\dagger | E \rangle . \quad (4.24)$$

Using the commutator (4.23), the rightmost expectation value is readily calculated to be $\delta^{(d)}(\vec{k}_1 + \vec{k}_2)$. This can be Fourier transformed to obtain the Wightman two-point function in position space [128]

$$G_E(\eta, \vec{x}; \eta' \vec{y}) \equiv \langle E | \hat{\phi}(\eta, \vec{x}) \hat{\phi}(\eta', \vec{y}) | E \rangle \\ \propto {}_2F_1 \left(\Delta_+, \Delta_-; \frac{d+1}{2}; \frac{1+P}{2} \right) , \quad (4.25)$$

which⁴ depends on the spacetime points only through the geodesic distance $P(\eta, \vec{x}; \eta', \vec{y})$ given in (4.12), and satisfies the equation of motion (4.15) in the way a Green’s function should. The dependence only on P indicates that the Euclidean vacuum is invariant under the de Sitter isometries. However, $G_E(P)$ is not the only solution to the equations of motion (4.15) that is de Sitter invariant. The function $G_E(-P)$ satisfies the same requirements, and therefore provides another valid scalar field Green’s function. This term contributes to the propagator of the scalar field in so called “ α -vacua”, due to Mottola and Allen [129, 130] (see also [141, 140] for discussions in the context of dS/CFT). These vacua are defined to be annihilated by the operators

$$\tilde{a}_{\vec{k}} = \frac{\hat{a}_{\vec{k}} - \alpha \hat{a}_{-\vec{k}}^\dagger}{\sqrt{1 - |\alpha|^2}} , \quad [\tilde{a}_{\vec{k}}^\dagger, \tilde{a}_{\vec{k}'}^\dagger] = \delta^{(d)}(\vec{k} - \vec{k}') , \quad \tilde{a}_{\vec{k}} |\alpha\rangle = 0 , \quad (4.26)$$

with $|\alpha| < 1$. The field operator (4.22) can be written in terms of these operators multiplied by mode functions of the form $\varphi_k + \bar{\alpha} \bar{\varphi}_k$. The states $|\alpha\rangle$ can be written as highly excited states built on the Euclidean vacuum. Using the Baker-Campbell-Hausdorff formula (2.28), one can show that

$$|\alpha\rangle \propto e^{\frac{\alpha}{2} \int d^3 k' a_{\vec{k}'}^\dagger a_{-\vec{k}'}^\dagger} |E\rangle . \quad (4.27)$$

⁴ If we restrict this result to the static patch and Wick rotate static patch time $t \rightarrow -it_E$, this propagator would continue to the unique Euclidean propagator on the sphere S^d , thus explaining the name “Euclidean vacuum” [140].

What makes the Euclidean propagator (4.25) unique among the α -vacua is its singularity structure. The hypergeometric function is singular⁵ at $P = 1$, whereas $G_E(-P)$ has a singularity at $P = -1$. As discussed at the end of §4.1, points with $P = 1$ are null separated. The propagators of other α -vacua have a singularity when $P = -1$, which is when one point is null separated from the antipodal reflection of the other point. The Euclidean vacuum is distinguished⁶ by the absence of this antipodal singularity.

4.2.3 The conformally coupled scalar in 3+1 dimensions

The case that will be of primary interest later on will be $d = 3$ and $m^2 = 2$. In this case, we have $\Delta_- = 1$ and $\Delta_+ = 2$. The Hankel functions in (4.16) and Bessel functions in (4.20) simplify considerably when $\nu = 1/2$,

$$\varphi_{\vec{k}} = \frac{i\eta}{\sqrt{2k}} e^{-ik\eta}, \quad \varphi_{(1,\vec{k})} = -\eta \cos(-k\eta), \quad \varphi_{(2,\vec{k})} = \frac{-\eta}{k} \sin(-k\eta). \quad (4.28)$$

Apart from an overall factor η , the positive and negative frequency modes $\varphi_{\vec{k}}$ and $\bar{\varphi}_{\vec{k}}$ of this scalar field oscillate as in flat space for all η . Furthermore, the Wightman function in the Euclidean vacuum is given by

$$\langle E | \hat{\phi}_{\vec{k}}(\eta) \hat{\phi}_{\vec{k}'}(\eta') | E \rangle = \frac{\eta\eta'}{2\sqrt{kk'}} e^{-ik\eta + ik'\eta'} \delta^{(3)}(\vec{k} + \vec{k}'). \quad (4.29)$$

To get the position space result, we can do the Fourier transform,

$$\begin{aligned} G_E(\vec{x}, \eta; \vec{y}, \eta') &= \frac{\eta\eta'}{2(2\pi)^3} \int d^3k \frac{1}{k} e^{ik(\eta' - \eta) + i\vec{k} \cdot (\vec{x} - \vec{y})} \\ &= \frac{\eta\eta'}{4\pi^2} \frac{1}{|\vec{x} - \vec{y}|} \int dk e^{ik(\eta' - \eta)} \sin(k|\vec{x} - \vec{y}|). \end{aligned} \quad (4.30)$$

However, this result does not converge at large k and needs to be regulated. Introducing a positive number ϵ , we can replace the exponent by $ik(\eta' - \eta + i\epsilon)$ to guarantee convergence of the result,

$$G_E(\vec{x}, \eta; \vec{y}, \eta') = \frac{1}{4\pi^2} \frac{\eta\eta'}{(\vec{x} - \vec{y})^2 - (\eta' - \eta + i\epsilon)^2}. \quad (4.31)$$

⁵ Along the light-cone, this propagator is sensitive to the $i\epsilon$ prescription. We will discuss this in more detail in §4.2.3.

⁶ A local observer in de Sitter space would not encounter this singularity. Consider any observer in de Sitter and choose the northern static patch, indicated in gray in Figure 4.2, to coincide with their causal diamond. The antipodal reflection of any point accessible to this observer is located in the southern static patch, and is spacelike separated from any point in the northern static patch.

Apart from the $i\epsilon$ term, the result agrees with (4.25) for $d = 3$ and $m^2 = 2$. However, when the two points are light-like separated, $(\vec{x} - \vec{y})^2 = (\eta' - \eta)^2$, the result would diverge but for the term $i\epsilon$. We can write the contribution using the Sokhotsky formula

$$\frac{1}{1 \pm i\epsilon} = \mathcal{P} \frac{1}{x} \mp i\pi \delta(x) , \quad (4.32)$$

where \mathcal{P} indicates the principal value of the expression. The $i\epsilon$ prescription thus leads to a contribution located on the light-cone,

$$\begin{aligned} G_E(\vec{x}, \eta; \vec{y}, \eta') &= \mathcal{P} \frac{1}{4\pi^2} \frac{\eta\eta'}{(\vec{x} - \vec{y})^2 - (\eta' - \eta)^2} \\ &+ \frac{i}{8\pi} \text{sign}(\eta' - \eta) \frac{\eta\eta'}{|\vec{x} - \vec{y}|} \delta(|\vec{x} - \vec{y}| - |\eta' - \eta|) , \end{aligned} \quad (4.33)$$

The appearance of this imaginary contribution can be traced back to the field theory analog of the typical quantum mechanics commutator $[\hat{x}, \hat{p}] = i$. This is most straightforwardly seen by calculating the momentum space commutator from (4.29),

$$\begin{aligned} \langle E | [\hat{\phi}_{\vec{k}}(\eta), \hat{\pi}_{\vec{k}'}(\eta)] | E \rangle &= \frac{1}{\eta^2} (\partial_{\eta'} - \partial_{\eta}) \langle E | \hat{\phi}_{\vec{k}}(\eta) \hat{\phi}_{\vec{k}'}(\eta') | E \rangle \Big|_{\eta' \rightarrow \eta} \\ &= i \delta^{(3)}(\vec{k} + \vec{k}') , \end{aligned} \quad (4.34)$$

and performing the Fourier transformation. Equivalently, one can act with the combination $\partial_{\eta'} - \partial_{\eta}$ on (4.31) and integrate over space, postponing the $\epsilon \rightarrow 0$ limit until the end.

There is another formulation of the field operator $\hat{\phi}$ which will be useful in the discussion of dS holography later on. We can rewrite the (4.22) in terms of the real modes,

$$\hat{\phi}(\eta, \vec{x}) = \int \frac{d^d k}{(2\pi)^{\frac{3}{2}}} \left(\hat{\alpha}_{\vec{k}} \varphi_{(1,k)} + \hat{\beta}_{\vec{k}} \varphi_{(2,k)} \right) e^{i\vec{k} \cdot \vec{x}} , \quad (4.35)$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the Hermitian operators

$$\hat{\alpha}_{\vec{k}} = \frac{i}{\sqrt{2k}} (\hat{a}_{-\vec{k}}^\dagger - \hat{a}_{\vec{k}}) , \quad \hat{\beta}_{\vec{k}} = \sqrt{\frac{k}{2}} (\hat{a}_{\vec{k}} + \hat{a}_{-\vec{k}}^\dagger) , \quad (4.36)$$

which satisfy the commutation relations $[\hat{\alpha}_{\vec{k}}, \hat{\beta}_{\vec{k}'}] = -i\delta^{(d)}(\vec{k} + \vec{k}')$. We will consider the states that are annihilated by these operators,

$$\hat{\alpha} |D\rangle = 0 , \quad \hat{\beta} |N\rangle = 0 . \quad (4.37)$$

The state $|D\rangle$ corresponds to the α -vacuum limit $\alpha \rightarrow 1$, whereas $|N\rangle$ is given by $\alpha \rightarrow -1$. These limits of (4.26) are singular, and in fact the norms of $|D\rangle$ and $|N\rangle$ are infinite. However, their overlap with the Euclidean vacuum is 1 and the overlap with finite excitations of $|E\rangle$ remains finite. It therefore makes sense to calculate the following two-point functions,

$$\langle D | \hat{\phi}_{\vec{k}}(\eta) \hat{\phi}_{\vec{k}'}(\eta') | E \rangle = \frac{i\eta\eta'}{2k} \sin(-k\eta) e^{ik\eta'} \delta^{(3)}(\vec{k} + \vec{k}') , \quad (4.38)$$

$$\langle N | \hat{\phi}_{\vec{k}}(\eta) \hat{\phi}_{\vec{k}'}(\eta') | E \rangle = -\frac{\eta\eta'}{2k} \cos(-k\eta) e^{ik\eta'} \delta^{(3)}(\vec{k} + \vec{k}') . \quad (4.39)$$

In the $\eta \rightarrow 0$ limit, the first two-point function contains only the fall-off $\sim \eta^2$, whereas the second only contains $\sim \eta$.⁷ It is these two-point functions, and not the Euclidean one, which are the dS analogs of the AdS two-point functions with one of the fall-offs fixed.

As a final remark, observe that the corresponding two-point functions in position space contain the antipodal singularity (from (4.12), observe that $P \rightarrow -P$ is achieved by $\eta \rightarrow -\eta$), which is characteristic of every vacuum but the Euclidean one

$$\begin{aligned} \langle D | \hat{\phi}(\eta, \vec{x}) \hat{\phi}(\eta', \vec{y}) | E \rangle = \frac{\eta\eta'}{8\pi^2} \left(\frac{1}{(\vec{x} - \vec{y})^2 - (\eta' - \eta + i\epsilon)^2} \right. \\ \left. - \frac{1}{(\vec{x} - \vec{y})^2 - (\eta' + \eta + i\epsilon)^2} \right) , \end{aligned} \quad (4.40)$$

$$\begin{aligned} \langle N | \hat{\phi}(\eta, \vec{x}) \hat{\phi}(\eta', \vec{y}) | E \rangle = -\frac{\eta\eta'}{8\pi^2} \left(\frac{1}{(\vec{x} - \vec{y})^2 - (\eta' - \eta + i\epsilon)^2} \right. \\ \left. + \frac{1}{(\vec{x} - \vec{y})^2 - (\eta' + \eta + i\epsilon)^2} \right) . \end{aligned} \quad (4.41)$$

4.3 Wave function for de Sitter Universes

So far, we have described field theory on dS using canonical quantization. Just as in quantum mechanics, this formulation is connected to other, equivalent representations of quantum field theory, such as the wave function formalism

⁷ Stripping off the overall factor of η , this was the inspiration for calling $|D\rangle$ the “Dirichlet” state and $|N\rangle$ the “Neumann” state: the two-point function (divided by η) with a Dirichlet boundary condition vanishes as $\eta \rightarrow 0$, whereas the two-point function (divided by η) with a Neumann boundary condition has a vanishing derivative as $\eta \rightarrow 0$.

and Feynman's path integral representation. These different formulations find their use in attempts to go beyond the description of fields on a fixed background and try to find a quantum theory for the Universe as a whole.

In [142, 143], Hartle and Hawking proposed the *no-boundary wave function* (NBWF) to describe the quantum state on a time-slice of a $3 + 1$ -dimensional spatially closed universe. An expression for this wave function was given in terms of a path integral over compact 4-geometries and field profiles, which end on the aforementioned 3-dimensional slice and have no other boundary, weighted by the exponent of the Euclidean action.

In this section, we will briefly describe the wave function and Feynman path integral description of §4.2.2, before we proceed to the proposal of [142, 143] for a wave function of the Universe.

4.3.1 States, wave functions and path integrals

We can associate a wave function to states $|\psi\rangle$ in a Hilbert space, by taking the inner product $\psi(q) = \langle q|\psi\rangle$ with the eigenstates $|q\rangle$ of a set of commuting Hermitian operators \hat{q}_i , if those eigenvectors span the full Hilbert space, and if the \hat{q}_i have a continuous spectrum. In the quantum mechanical description of a point particle, q_i can be the coordinates of the particle's position and the index i is discrete. In the quantum field theory of §4.2.2, we can use the operators $\phi(\eta, \vec{x})$ on a fixed η slice as the set of commuting Hermitian operators. The role of the index i is then played by continuous vector \vec{x} .

It is possible to write certain wave functions as a path integral. As an illustration [142], consider a bound particle in quantum mechanics with Hamiltonian $H(q, p) = \langle q|\hat{H}|p\rangle$ and a discrete set of eigenstates $|n\rangle$ with energies E_n , with $E_0 = 0$. The propagation of the particle from position 0 to position q in time t can be expressed as a path integral [144],

$$\langle q|e^{-i\hat{H}t}|0\rangle = \int_{x(0)=0}^{x(t)=q} \mathcal{D}x(t') \mathcal{D}p(t') e^{i \int [p dx - H(x, p) dt']} , \quad (4.42)$$

We can use the complete set of energy eigenstates to express the left-hand side as

$$\sum_n e^{-iE_n t} \langle q|n\rangle \langle n|0\rangle = \sum_n e^{-iE_n t} \psi_n(q) \bar{\psi}_n(0) . \quad (4.43)$$

If we Wick rotate this expression $t \rightarrow -i\tau$, the coefficient of each energy eigenstate is weighted by a factor that is exponential in its energy. In the limit

$\tau \rightarrow \infty$ only the ground state contributes. We can therefore express the ground state wave function as the path integral (if $\bar{\psi}_0(0) \neq 0$),

$$\begin{aligned} \psi_0(q) &= \frac{1}{\bar{\psi}_0(0)} \lim_{\tau \rightarrow \infty} \sum_n e^{-E_n \tau} \langle q|n\rangle \langle n|0\rangle \\ &\propto \lim_{\tau \rightarrow \infty} \int_{x(-\tau)=0}^{x(0)=q} \mathcal{D}x(\tau') \mathcal{D}p(\tau') e^{i \int p dx - \int H(x,p) d\tau'} . \end{aligned} \quad (4.44)$$

This path integral can be approximated by the steepest descent approximation when the on-shell action is large (with respect to $\hbar = 1$). The dominant contribution then comes from the solutions to the Euclidean (Wick rotated) equations of motion which go to zero in the limit $\tau \rightarrow -\infty$.

This representation can be extended⁸ to quantum field theory. If the steepest descent approximation is valid, the dominant contribution to the wave function comes from the on-shell Euclidean action (given by $S = -iS_E|_{t \rightarrow -i\tau}$),

$$\psi[\eta_0, \phi_0] \propto e^{-S_E[\eta_0, \phi]} , \quad (4.45)$$

where ϕ is the solution to the Euclidean equations of motion that remains finite for $\eta \rightarrow i\infty$ (the “Euclidean boundary conditions”) and which matches ϕ_0 at time η_0 . As an example [20], consider the free massless scalar field in dS_{3+1} . The classical solutions which remain finite at $\eta \rightarrow i\infty$ are the modes $\bar{\varphi}_{\vec{k}}$ with $\nu = 3/2$ given in (4.16). Normalizing these modes such that they have fixed Fourier modes $\phi_{0,\vec{k}}$ at time η_0 ,

$$\phi(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \phi_{0,\vec{k}} \frac{(i + k\eta)e^{ik\eta}}{(i + k\eta_0)e^{ik\eta_0}} . \quad (4.46)$$

The action (4.13) evaluated on these solutions can be calculated by performing integration by parts and using the equations of motion. The result is the boundary term

$$S_E[\eta_0, \phi] = \frac{1}{2} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{k^2}{\eta_0(1 - ik\eta_0)} \phi_{0,-\vec{k}} \phi_{0,\vec{k}} . \quad (4.47)$$

Using (4.45), this gives a *Gaussian* wave function of each of the $\phi_{0,\vec{k}}$. As in quantum mechanics, the square of the wave function provides a probability density that can be used to calculate expectation values. The two-point function

$$\langle E | \hat{\phi}_{\vec{k}}(\eta) \hat{\phi}_{\vec{k}'}(\eta') | E \rangle = \int \mathcal{D}\phi \phi_{\vec{k}} \phi_{\vec{k}'} |\Psi(\phi)|^2 = \frac{1 + k^2 \eta_0^2}{2k^3} \delta^{(3)}(\vec{k} + \vec{k}') . \quad (4.48)$$

⁸ At least formally, the path integral gives a representation of certain wave functions. However, it is not guaranteed to converge in general and may require regularization and renormalization.

corresponds to the result in (4.24) for $\nu = 3/2$. Indeed, given the definition of the wave function, this is just the two-point function written as a path integral with two insertions of the field and “Euclidean boundary” conditions in the past and future.

4.3.2 The no-boundary wave function

When gravity is included, the geometry of spacetime becomes a fluctuating quantum field. The geometry of a three-dimensional space becomes an argument of the wave function. This can be characterized by a three-metric h_{ij} . In order to write it as a path integral, this three-dimensional slice should be embedded in a four-dimensional spacetime, the metric of which is integrated over.

In a theory for gravity in a spacetime with closed spatial slices, there is an interesting new alternative to (4.44): the four-geometry that connects to h_{ij} on one side, can smoothly cap off and not have another boundary. This idea was used in [142] to propose a specific “no-boundary” wave function

$$\psi_{\text{HH}}[h_{ij}, \chi] \propto \int_{\mathcal{C}} \mathcal{D}g_{\mu\nu} \mathcal{D}\phi e^{-S_E[g, \phi]}, \quad (4.49)$$

where $g_{\mu\nu}$ and ϕ denote the four-metric and matter fields integrated over the set \mathcal{C} all *compact* four-geometries with a unique boundary for which the geometry is specified by h_{ij} . The exponent S_E is the Euclidean Einstein-Hilbert plus matter action

$$S_E = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{g} (R - 2\Lambda + \mathcal{L}_{\text{matter}}) + \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{h} K. \quad (4.50)$$

This wave function formally satisfies the Wheeler-deWitt equation [145], which takes the form of a time-independent Schrödinger equation

$$\hat{H}\psi_{\text{HH}}[h_{ij}, \chi] = 0, \quad (4.51)$$

where H is the Hamiltonian of the ADM formalism for GR [146] with momenta replaced by functional derivatives with respect to the three-metric h_{ij} . This Hamiltonian is obtained as the Legendre transformation of (4.50). We will give the explicit form of this Hamiltonian in (5.15). The Wheeler-deWitt equation expresses that the wave function does not depend on the coordinate time t , since that is not a gauge-invariant quantity. Indeed, no reference was made to any explicit time in the definition (4.49).

Global de Sitter is a spacetime with compact spatial slices, so the wave function proposal can be applied to Einstein gravity with a positive cosmological

constant. This case was already considered in [142, 143] in a “minisuperspace model”. This means that the line element of the four-geometry can be written as $ds^2 = N^2(\tau)d\tau^2 + a(\tau)^2 d\Omega_3^2$, and only the N and the scale factor a are integrated over in the path integral. In the semi-classical (steepest descent) approximation, the resulting wave function $\psi(a_0)$ (where a_0 is the boundary value of the scale factor) was found to be exponentially small for scale factors $a_0 \ll l$ much smaller than the de Sitter length, whereas it oscillates as a cosine of $(a_0/l)^3$ for $a_0 \gg l$.

Outside of minisuperspace, the no-boundary wave function on asymptotically de Sitter spaces is a function of more general metric h_{ij} and matter fields, for which we will use a scalar field ϕ . Such spacetimes can be written using the Fefferman-Graham expansion, which takes a form similar to that in asymptotically AdS,

$$ds^2 = -\frac{d\eta^2}{\eta^2} + h_{ij}(\eta, x)dx^i dx^j, \quad h_{ij}(\eta, \vec{x}) = \frac{1}{\eta^2}[\gamma_{ij}(\vec{x}) + \mathcal{O}(\eta^2)], \quad (4.52)$$

where $\gamma_{ij}(x)$ is constant. Furthermore, the scalar field has the same behavior as before in the $\eta \rightarrow 0$ limit, with two modes $\alpha(x)\eta^{\Delta_-}$ and $\beta(x)\eta^{\Delta_+}$. At finite η , these coefficients become time-dependent,

$$\chi(\eta, \vec{x}) = \eta^{\Delta_-}[\alpha(x) + \mathcal{O}(\eta^2)] + \eta^{\Delta_+}[\beta(x) + \mathcal{O}(\eta^2)]. \quad (4.53)$$

Given a wave function of a universe, an interesting question one could ask is whether such a wave function describes a universe evolving according to the classical equations of motion (to good approximation). This question was answered in [147] by considering the semi-classical contributions to the wave function. If the on-shell action \mathcal{S}_E , as a function of the boundary conditions of the path integral, has a more rapidly varying imaginary part $\mathcal{S} \equiv \text{Im}(\mathcal{S}_E)$ than real part $I_R \equiv \text{Re}(\mathcal{S}_E)$,

$$|\vec{\nabla} I_R| \ll |\vec{\nabla} \mathcal{S}|, \quad (4.54)$$

then \mathcal{S} provides a Hamilton-Jacobi generating function of the solutions to the classical equations of motion. The time-evolution of this semi-classical contribution to the wave function is then well-approximated by the classical time evolution. For this reason, (4.54) are called *classicality conditions* [147].

In Chapter 5, we will consider the no-boundary wave function and classicality conditions in much more detail. We are especially interested in the wave function in the (γ_{ij}, α) basis, instead of (h_{ij}, χ) . This is motivated by the dS/CFT proposal, which we will now review.

4.4 dS/CFT

Even if a full UV complete theory of quantum gravity is not currently understood, we can try to find an exact and tractable description of the subsector of asymptotically de Sitter universes. As outlined in the previous chapters, there is such a proposal for quantum gravity with a negative cosmological constant. From our review of the (A)dS geometry and scalar field theory, it is clear that there are many similarities, in particular near their respective conformal boundaries. The question arises to what extent a framework similar to AdS/CFT can be set up for universes with a positive cosmological constant, more precisely for spacetimes which are asymptotically de Sitter.

Such a framework was named dS/CFT in [18], which developed some of its foundations. Quantum gravity in a universe with an asymptotically dS_{d+1} boundary was conjectured to be related to a Euclidean CFT on S^d . On the planar patch, the dual CFT is defined on \mathbb{R}^d .

As a first consistency check at the level of symmetries, the isometry generators of de Sitter indeed satisfy the same commutation relations (4.2) as the Euclidean conformal generators (2.19). For example in the planar patch, the conformal generators (2.25), (2.27), (2.30) and (2.31) are represented as differential operators on the bulk fields

$$\begin{aligned} \hat{P}_i &\leftrightarrow \partial_i, & \hat{M}_{ij} &\leftrightarrow x_j \partial_i - x_i \partial_j, \\ \hat{D} &\leftrightarrow x^i \partial_{x^i} + \eta \partial_\eta, & \hat{K}_i &\leftrightarrow x_i (x^j \partial_{x^j} + \eta \partial_\eta) + 2(\eta^2 - x^2) \partial_{x^i}. \end{aligned} \quad (4.55)$$

The proposal of [18] is that the CFT correlation functions can be calculated in dS as correlation functions of fields in dS with certain boundary conditions on \mathcal{I}^\pm . The simplest example is the planar patch scalar two-point function. Using (4.12), one can see that the Euclidean propagator (4.25) function contains the two-point functions of a scalar primary with weight $\Delta_\pm = \frac{d}{2} \pm \nu$, with $\nu = \sqrt{d^2 - 4m^2}/2$,

$$G_E(\eta, \vec{x}; \eta', \vec{y}) \approx c_{\Delta_-} \left(\frac{\eta \eta'}{(x_1 - x_2)^2} \right)^{\Delta_-} + c_{\Delta_+} \left(\frac{\eta \eta'}{(x_1 - x_2)^2} \right)^{\Delta_+},$$

where c_{Δ_\pm} are constants that depend on the weights and dimension d . Choosing different asymptotic boundary conditions on the fields, either of these terms can be isolated to get a CFT-like two-point function. The relation between the bulk scalar mass and the conformal weights in the CFT are

$$\Delta(\Delta - d) = -m^2. \quad (4.56)$$

As was observed in [18] the minus sign relative to corresponding entry in the AdS dictionary implies that bulk scalar fields with mass $m^2 > d^2/4$, the relevant quantity is imaginary and the CFT is not unitary. By definition, that means that the CFT is not the Wick rotation of a Lorentzian theory with a probabilistic interpretation. However, as was already mentioned in [18], there seems to be no obvious reason within the dS/CFT proposal for it to be so.

The correlation functions of CFT operators of spin s are similarly given in terms of bulk fields of spin s . In particular, the CFT stress tensor T_{ij} is related to the bulk metric h_{ij} .

The relation between bulk and CFT correlation functions can be expressed in terms of the corresponding generating functions of correlations [19, 20],

$$Z_{\text{CFT}}[\chi, h_{ij}] = \Psi[\chi, h_{ij}] , \quad (4.57)$$

where Ψ is defined as a path integral as in §4.3. In [20], this generating function was interpreted as the path integral expression of the Hartle-Hawking wave function (4.49). The generating function Z_{CFT} can be used to derive CFT correlation functions of primary operators as in (2.35). The currents are γ_{ij} and α as given in (4.52) and (4.53).

In the next chapters, we will analyze this dS/CFT proposal in more detail. We will analyze Ψ in de Sitter space in Chapter 5. We will calculate without reference to the CFT partition function, only using the path-integral expression (4.49). Nevertheless, inspired by (4.57), we will express the wave function in terms of α in (4.53). From Chapter 6 onward, we will proceed with a model that realizes the dS/CFT proposal. We first conclude this chapter with an outline of the differences between holography in AdS and in dS.

4.5 Comparing dS/CFT and AdS/CFT

The framework of dS/CFT was inspired by AdS/CFT, and is formulated in very similar terms. Nevertheless, a Euclidean CFT describes fundamentally different phenomena than a Lorentzian one and the physics of a dS universe differs dramatically from that in AdS. Already at the classical level, the presence of observer-dependent cosmological horizons in dS complicates the causal structure of the spacetime. The basic notion of observability is obfuscated by the fact that no observer in de Sitter has access to all events. This is sharply contrasted by the test tube-like nature of AdS where all spatial points can be probed within a finite amount of time, even if they are an infinite distance away.

Semi-classically, the well-controlled perturbative calculations in AdS are a lot more straightforward than in de Sitter space, where already the choice of vacuum state is subtle, owing to the lack of globally well-defined notion of energy. Furthermore, semi-classical calculations show that there is a temperature associated with the cosmological horizons in dS. Universes with a positive cosmological constant appear to be thermal equilibrium states, with nonzero values of thermodynamic quantities such as temperature and entropy.

We will therefore dedicate this section to collecting several aspects that set holography in the cosmological case apart from the “test tube” case of AdS/CFT. Each of these observations relates to the main conclusion of this section, which is formulated in §4.5.4: an essential structure which is absent in the current dS/CFT proposal is a bulk Hilbert space.

4.5.1 The role of time

The most obvious and arguably the most important difference between the two holographic proposals is the role played by the time direction. In AdS/CFT, the CFT time coincides with time in the Fefferman-Graham (FG) expansion in asymptotically AdS. There is a timelike asymptotic Killing vector field which runs parallel to the conformal boundary, and which corresponds to CFT time translations. The associated symmetry generators are related as well. In the CFT this is the ordinary Hamiltonian, whereas its bulk equivalent can be used to define a quasilocal notion of energy [116].

This is a very useful aspect of the correspondence. It enables one to compare the possible (perturbative) spectra of fields and thermodynamic quantities on both sides of the correspondence, both of which were important arguments for the more general framework of AdS/CFT. Furthermore, the CFT dynamics is directly related to the dynamics in AdS, since it sets the boundary conditions at the conformal boundary.

This entire structure is absent in dS/CFT, where the CFT is defined on a Riemannian manifold and time must emerge holographically. There is no CFT dynamics that could be related to bulk time evolution, nor is there a symmetry generator that we can unambiguously identify to be the Hamiltonian. Even if a local CFT Lagrangian exists, it is not related to a Hamiltonian by the usual Legendre transformation. Indeed, that would require the identification of a special “time direction” with respect to which canonical momenta could be defined.

The situation is not much better on the bulk side. There is no Killing vector which is timelike everywhere, and therefore there are no conserved charges which

are positive everywhere. Even close to \mathcal{I}^\pm , there is no Killing vector field which is everywhere timelike. Global de Sitter time translations are not isometries, nor do they become approximate isometries near the conformal boundary.

The planar patch relations (4.55) indicate that time should emerge from the CFT in a manner related to scale/conformal transformations, such that it corresponds to the asymptotic scaling symmetry at late times in the planar patch.⁹ In that sense, bulk time should be related to the renormalization group flow of the CFT. Using (4.57), this suggests that time evolution of the dS wave function is given by the RG evolution of the CFT partition function.¹⁰ However, essential elements of this correspondence remain unclear. For instance, it is not obvious which precise property in the CFT corresponds to unitarity of time evolution of the dS wave function.

We will try to make progress in Chapter 7, by identifying the particular combination of CFT data that corresponds to spacetime localized data in otherwise undisturbed de Sitter space.

4.5.2 Boundary conditions

As we have explained, the classical Cauchy problem in AdS requires the specification of asymptotic boundary conditions. (In the Poincaré patch, this must be supplemented with regularity conditions in the bulk.) In AdS/CFT, the boundary conditions are specified by the CFT sources as in (2.46). Thus, only the subleading fall-off modes of the fields becomes a quantized degree of freedom as a CFT operator, whereas the other is fixed.

In dS/CFT, the analogous boundary conditions are in the far past and future. (In the planar patch, the analog of regularity conditions are the Euclidean boundary conditions.) This contrasts the classical Cauchy problem, which only requires initial conditions. Similarly in canonical quantization, both fall-off modes of the scalar field (4.53) are quantized, whereas the bulk correlation functions proposed to be calculated by the CFT [18] are those of either $\hat{\alpha}$ or $\hat{\beta}$. Moreover, the bulk operators have the commutation relations of the Heisenberg algebra, whereas local operators in a Euclidean CFT commute. This is explicit when the CFT correlation functions are given in terms of a Euclidean path integral, which will be the case for the model described in Chapter 6. In Chapter 7, we will make this observation more precise.

⁹ Planar patch time translations are generated by a conformal Killing vector field. They do not correspond to an isometry.

¹⁰ These ideas have been made more precise in AdS [148].

4.5.3 The wave function interpretation

The dS/CFT relation (4.57) conjectures that the CFT partition function provides a UV completion of the no-boundary wave function (4.49). However, it does not provide an integration measure over which must be integrated to calculate probabilities. Without an integration measure, this partition function cannot be said to be “normalizable” and it is not evident that it can be interpreted as a wave function.

4.5.4 CFT Hilbert space structure

In the known examples of AdS/CFT, there is a Hilbert space, so that probabilities and expectation values can be defined in the usual way and time evolution is unitary. According to the conjecture, the Hilbert space of quantum gravity is that of the CFT. The CFT operators obey the unitarity bounds formulated in §2.3.4, which are necessary to guarantee positivity of the inner product.

The situation is radically different for CFT’s that are holographically dual to dS. There is no CFT notion of time evolution with conserved probabilities. The CFT is not required to be reflection positive. Indeed, one can see from (4.48) that if empty de Sitter space is to be a local maximum of the bulk wave function, the CFT two-point function of any operator in the vacuum must be negative. This will indeed be the case for the model presented in Chapter 6. The absence of a positive definite inner product implies that the CFT does not have the usual Hilbert space description with a probabilistic interpretation. In the context of (4.57), it is therefore not clear to what extent we can interpret the CFT partition function as a wave function in the usual sense, namely one that is normalized with respect to some inner product and from which probabilities can be derived.

In summary, it is not clear whether the dS/CFT proposal provides the Hilbert space of quantum gravity in dS.

These remarks set the course for the remainder of this thesis. As outlined in the introduction, we will first consider the no-boundary path integral in more detail in Chapter 5. In particular, we analyze the wave function in the basis of CFT quantities such as α in (4.53). In Chapter 6 we will review models which realize the dS/CFT proposal and review to what level they have been analyzed in the literature. In Chapter 7, we analyze if dS/CFT provides a generalization of the theory set up in §4.2.2, in which case the properties of a Hilbert space should appear in the perturbative regime. In particular, we

will look for CFT operators which transform under the conformal group as local field operators $\hat{\phi}(x)$ in the bulk. This clarifies how bulk time evolution is encoded in the CFT. We will find that not all bulk correlation functions can be reproduced by the CFT, essentially because the Euclidean path integral does not give rise to commutators of operators, such as $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^{(d)}(x - y)$.

Starting in Chapter 8, we will outline a proposal that aims to address these problems. Our starting point is the construction of a Hilbert space for the higher-spin gravitational theory in the model of Chapter 6.

Chapter 5

Lorentzian condition in holographic cosmology [2]

This chapter is a reprint of [2] where we derived a sufficient set of conditions on the Euclidean boundary theory in dS/CFT for it to predict classical, Lorentzian bulk evolution at large spatial volumes. Our derivation makes use of a canonical transformation to express the bulk wave function at large volume in terms of the sources of the dual partition function. This enables a sharper formulation of dS/CFT. The conditions under which the boundary theory predicts classical bulk evolution are stronger than the criteria usually employed in quantum cosmology. We illustrate this in a homogeneous isotropic minisuperspace model of gravity coupled to a scalar field in which we identify the ensemble of classical histories explicitly.

5.1 Introduction

The dS/CFT correspondence [149, 18, 20, 19] conjectures that the wave function of the universe with asymptotic de Sitter (dS) boundary conditions is given in terms of the partition function of a Euclidean CFT deformed by various operators. Explicit realizations of dS/CFT include the duality between Vasiliev gravity in dS and Euclidean $Sp(N)$ vector models [21] and, at the semiclassical level, the holographic form of the Hartle-Hawking wave function put forward in [150], building on earlier work [20, 151, 152, 153, 154] and further explored in [155, 156, 27, 157, 158, 159].

In dS/CFT the arguments of the wave function of the universe are related to external sources in the dual partition function. The dependence of the partition function on the values of these sources thus yields a holographic measure on the space of asymptotically locally de Sitter configurations. But it has remained an open question what is the exact configuration space of deformations on which the holographic wave function in dS/CFT should be defined¹. Here we study a particular constraint on this configuration space that arises from the condition that the holographic wave function must predict classical Lorentzian spacetime evolution in the bulk at least for large spatial volumes.

The classical behavior of geometry and fields on sufficiently large scales follows from the Wheeler-DeWitt (WDW) equation at large volume and thus applies to any wave function satisfying the Hamiltonian constraint. It is usually said that a wave function of the universe predicts classical evolution if its phase varies rapidly compared to its amplitude in all directions in superspace. This criterion is known as the classicality condition and is analogous to the prediction of the classical behavior of a particle in a WKB state in non-relativistic quantum mechanics².

However this derivation of classical evolution does not carry over to the dual partition functions featuring in dS/CFT. This is because the large phase factor of the bulk wave function is absent in the partition function. Specifically it is canceled by the addition of counterterms. This raises the question whether the emergence of classical Lorentzian evolution in the bulk for large spatial volumes can be identified and understood from the dual partition function.

In this paper we derive a sharper set of classicality conditions that are meaningful from an asymptotic viewpoint and which, in particular, can be applied to the Euclidean boundary theory in dS/CFT to verify whether it predicts classical Lorentzian bulk evolution in the large volume limit. Our derivation makes use of a canonical transformation to coordinates on superspace that are well-defined from a boundary viewpoint, namely the sources in the dual partition function. After a brief review of the Hartle–Hawking wave function [142] and its holographic form [150] in Section 5.2 we perform this transformation on the wave function in Section 5.3 to derive a new wave function that is a function of asymptotically well-defined superspace variables. The relation between the two wave functions resembles and generalizes the Fourier transformation between a wave function in momentum and position space.

The new wave function no longer contains a phase factor that grows with the spatial three-volume. This is in line with the results of a similar calculation

¹ For recent work on this see e.g. [27, 157, 160].

² See e.g. [161, 147] for a discussion of this in the context of quantum cosmology.

in the context of AdS holography in [125], where variables on phase space were identified that make the variational problem well-defined. It was found there that these variables are related to the original AdS fields and momenta by a canonical transformation, and that this transformation is equivalent to holographic renormalization. Specifically, the local boundary terms that regularize the AdS action are exactly the generating function of this canonical transformation. Similarly in the de Sitter context which we consider here, the generating function absorbs or ‘regularizes’ the local phase factor of the wave function.

The expression of the bulk wave function in terms of the sources of the dual enables a sharper and more appealing formulation of dS/CFT in which the wave function of the universe is directly related to the partition functions of deformed Euclidean CFTs. In Section 5.4 we revisit the question of classicality in this context. We analyze the conditions under which the new wave function predicts classical bulk evolution at large spatial volumes and interpret these from a dual viewpoint as the requirement that the vevs must be approximately real. This is a stronger condition than the criterion for classical behavior usually employed in quantum cosmology which, in dual terms, involves the sources only. We illustrate this difference in Section 5.5 in a minisuperspace model where we identify the ensemble of classical histories explicitly.

We perform our calculations in the Hartle-Hawking state because dS/CFT is best developed in this context. However, most of our results apply more generally.

5.2 Holographic No-Boundary Measure

This section reviews the holographic formulation of the no-boundary wave function (NBWF) [142] put forward in [150], building on earlier work. The holographic form of the NBWF provides a concrete semiclassical realization of dS/CFT with which we work in the remainder of this paper.

5.2.1 No-Boundary Wave Function

A quantum state of the universe is specified at low energies by a wave function Ψ on the superspace of three-geometries $h_{ij}(\vec{x})$ and matter field configurations $\chi(\vec{x})$ on a closed spacelike three-surface Σ . The NBWF is given by a sum \mathcal{C} over regular four-geometries g and fields ϕ on a four-manifold M with one boundary Σ , weighted by $\exp(-I_E[g, \phi]/\hbar)$ where $I_E[g, \phi]$ is the Euclidean

action. Schematically, we write

$$\Psi(h_{ij}, \chi) = \int_{\mathcal{C}} \mathcal{D}g \mathcal{D}\phi \exp(-I_E[g(x), \phi(x)]/\hbar) . \quad (5.1)$$

Here, $g(x)$ (short for $g_{\alpha\beta}(x^\gamma)$) and $\phi(x)$ are the histories of the 4-geometry and matter field.

To analyze classical bulk evolution from a boundary viewpoint we will work with a toy model consisting of Einstein gravity coupled to a single scalar field, with action

$$I_E = - \int d^4x \sqrt{g} \left[\frac{1}{2\kappa} (R - 2\Lambda) - \frac{1}{2} (\nabla\phi)^2 - V(\phi) \right] + \frac{1}{\kappa} \int d^3x \sqrt{h} K , \quad (5.2)$$

where $\kappa = 8\pi G_N$ and h, K are the induced metric and extrinsic curvature on Σ . The integration in (5.1) is carried out along a suitable complex contour which ensures the convergence of (5.1) and the reality of the result. In this paper we concentrate on models in which the cosmological constant Λ and the potential V in the action (5.2) are positive. We further assume the potential $V(\phi)$ is quadratic with mass m^2 around a minimum at $\phi = 0$, where it vanishes.

5.2.2 Lorentzian Bulk Evolution

In the large three-volume region of superspace, the path integral (5.1) defining the NBWF can be approximated by the method of steepest descent [158]. In this regime the NBWF is approximately given by a sum of terms of the form

$$\Psi[h_{ij}, \chi] \approx \exp\{(-I_R[h, \chi] + iS[h, \chi])/\hbar\}. \quad (5.3)$$

Here $I_R[h_{ij}, \chi]$ and $-S[h_{ij}, \chi]$ are the real and imaginary parts of the Euclidean action $I_E[h_{ij}, \chi]$ of a saddle point history (g, ϕ) on a compact 4-disk M with one boundary Σ . Metric and field match the real values (h_{ij}, χ) on Σ and are otherwise regular on the disk, rendering (g, ϕ) generally complex in the interior.

In the large volume regime boundary configurations (h_{ij}, χ) evolve classically, according to the Lorentzian Einstein equation, because the NBWF oscillates and obeys the classicality conditions that say that its phase S varies rapidly compared to I_R [147],

$$|\vec{\nabla} I_R| \ll |\vec{\nabla} S| . \quad (5.4)$$

This is analogous to the prediction of the classical behavior of a particle in a WKB state in non-relativistic quantum mechanics. Thus the NBWF predicts an ensemble of classical, asymptotically de Sitter histories that are the integral

curves of S in superspace, with relative probabilities that are proportional to $\exp[-2I_R(h_{ij}, \chi)]$ and conserved under time evolution [147].

Integral curves are defined by integrating the classical relations relating the momenta $\pi_{(h)ij}(\vec{x})$ and $\pi_{(\chi)}(\vec{x})$ to derivatives of the action

$$\pi_{(h)ij}(\vec{x}) = \delta S / \delta h_{ij}(\vec{x}), \quad \pi_{(\chi)}(\vec{x}) = \delta S / \delta \chi(\vec{x}). \quad (5.5)$$

The momenta are proportional to the time derivatives of h_{ij} and χ . Thus the solutions $h_{ij}(\vec{x}, t)$ and $\chi(\vec{x}, t)$ of (5.5) define field histories $\hat{\phi}(x, t) \equiv \chi(x, t)$ and Lorentzian four-geometries $\hat{g}_{\alpha\beta}(x, t)$ by

$$ds^2 = -dt^2 + h_{ij}(x, t) dx^i dx^j \equiv \hat{g}_{\alpha\beta}(x, t) dx^\alpha dx^\beta, \quad (5.6)$$

in a simple choice of gauge. The real, classical, Lorentzian histories predicted by the NBWF are therefore not the same as the complex saddle points that determine their probabilities. Further, the relations between superspace coordinates and momenta (5.5) mean that to leading order in \hbar , and at any one time, the predicted classical histories do not fill classical phase space. Rather, they lie on a surface within classical phase space of half its dimension.

5.2.3 Holographic No-Boundary Measure

By exploiting the complex structure of the no-boundary saddle points one can derive a holographic form of the tree level no-boundary measure [150]. To see this we write the saddle point geometries as

$$ds^2 = (N^2 + N_i N^i) d\lambda^2 + 2N_i dx^i d\lambda + g_{ij} dx^i dx^j, \quad (5.7)$$

and introduce the complex time coordinate $d\tau = d\lambda N(\lambda)$. The action (5.2) of a saddle point history then includes an integral over time τ . Different contours for this time integral give different geometric representations of the saddle point, each giving the same amplitude for the boundary configuration (h_{ij}, χ) . This freedom in the choice of contour gives physical meaning to a process of analytic continuation – not of the Lorentzian histories themselves – but of the saddle points that define their probabilities.

In [150] this freedom of choice of contour was used to identify two different useful representations of the general saddle points corresponding to asymptotically de Sitter universes. In one representation (dS) the interior saddle point geometry behaves as if Λ and V were positive, and converges towards a real Lorentzian solution that is asymptotically dS. In the other (AdS) the Euclidean part of the interior geometry behaves as if these quantities were negative, and specifies a regular AdS domain wall. Asymptotically Lorentzian

de Sitter (dS) universes and Euclidean anti-de Sitter (AdS) spaces are thereby connected in the wave function. This connection can be made explicit using the asymptotic form of the saddle point solutions. If we define the variable η by

$$\eta(\tau) = i\eta_0 e^{iH\tau} = i\eta_0 e^{-Hy + iHx}, \quad (5.8)$$

with $H \equiv \sqrt{\Lambda/3}$ and η_0 an arbitrary scale that we will fix below, then the large volume expansion of the general complex solution of the Einstein equation can be written as

$$g_{ij} = \frac{1}{\eta^2} \left(\gamma_{ij} + \eta^2 \gamma_{(2)ij} + \eta^{3-\sigma} (\gamma_{(-)ij} + \eta \gamma_{(2-)ij} + \dots) \right. \\ \left. + \eta^3 \gamma_{(3)ij} + \eta^{3+\sigma} \gamma_{(+)ij} + \mathcal{O}(\eta^4) \right), \quad (5.9a)$$

$$\phi(\eta) = \eta^{\lambda_-} \gamma^{\frac{\lambda_-}{2\sigma}} (\alpha + \eta \alpha_{(1)} + \dots) \\ - \frac{\eta^{\lambda_+}}{\sigma} \gamma^{-\frac{\lambda_+}{2\sigma}} (\beta + \eta \beta_{(1)} + \dots) + \mathcal{O}(\eta^{\lambda_-+1}), \quad (5.9b)$$

where $\lambda_{\pm} \equiv 3/2(1 \pm \sqrt{1 - 4m^2/9H^2})$, $\sigma \equiv \lambda_+ - \lambda_-$ and γ is the determinant of γ_{ij} .

The asymptotic solutions are specified by the asymptotic equations in terms of the boundary functions γ_{ij} and α , up to the η^0 term in (5.9a) and to order η^{λ_+} in (5.9b). Beyond this the interior dynamics and the boundary condition of regularity on \mathcal{M} become important.

In asymptotically dS saddle points the phases of the fields at the South Pole (SP) – the center of the 4-disk \mathcal{M} – are tuned so that g_{ij} and ϕ become real for small η along a vertical line $x = x_{TP}$ in the complex τ -plane. Equations (5.9) show that along this curve the complex saddle point tends to a real, asymptotically dS history. However since the expansions are analytic functions of η there is an alternative asymptotically vertical contour located at $x_A = x_{TP} - \pi/2H$ along which the metric g_{ij} is also real, but with the opposite signature. Along this contour the saddle point geometry (5.9a) is asymptotically Euclidean AdS. Hence a contour which first runs along the $x = x_A$ line and then cuts horizontally to the endpoint $\tau = v$ provides a representation of the saddle points in which their interior geometry consists of a regular Euclidean AdS domain wall that makes a smooth transition to an asymptotically dS universe.

Figure 5.1 illustrates this for an $O(4)$ invariant saddle point in which the scalar field $\phi(0)$ at the SP at $\tau = 0$ is significantly displaced from the minimum of its potential. The symmetry allows one to identify both contours explicitly (as opposed to only asymptotically for general saddle points). The dS contour

first runs along the real axis to a turning point (TP) at $x_{TP} = \pi/2V(\phi(0))$ and then vertically to the endpoint v . It is at this point that the values of the metric and scalar field should correspond with the arguments (h_{ij}, χ) of the wave function³. This corresponds to the usual saddle point representation where a deformed four-sphere is smoothly joined onto an approximately real, inflationary universe in which the scalar slowly rolls down to its final value.

The AdS contour starts vertically but gradually moves away from the dS contour to $x = x_A$ at large y . Along this part of the contour the saddle point is an asymptotically AdS, spherically symmetric domain wall with a complex scalar field profile in the radial direction y . The complex transition region along the horizontal branch of the contour smoothly interpolates between the AdS and the dS domain. This contour has the same endpoint v , the same action, and makes the same predictions, but the saddle point geometry is different.

The real part of the Euclidean action along $x = x_A$ has the usual AdS divergences for large y . However, along the $x = x_{TP}$ curve the real part of the action is asymptotically constant and hence does not grow parametrically with y . Therefore, the horizontal branch of the AdS contour regulates the divergences of the AdS action.

It can be shown [150] that for general saddle points the divergent terms in the action of the horizontal part are precisely the regulating counterterms plus a universal phase factor S_{ct} . Moreover the action integral along the horizontal branch of the AdS contour does not contribute to the amplitude in the large y limit [150]. This means that the probabilities for all Lorentzian asymptotically dS histories in the NBWF are fully specified by the regularized action of the interior asymptotic AdS regime of the saddle points. Specifically,

$$I_E[\eta(v), h_{ij}, \chi] = -I_{aAdS}^{\text{reg}}[\bar{\gamma}_{ij}, \bar{\alpha}] + iS_{ct}[\eta(v), h_{ij}, \alpha] + \mathcal{O}(\eta(v)) . \quad (5.10)$$

Here I_{aAdS}^{reg} is the $y \rightarrow \infty$ limit of the regulated asymptotic AdS action. Given $(\eta(v), h_{ij}, \chi)$, one can calculate the barred quantities $\bar{\alpha}$ and $\bar{\gamma}_{ij}$ as the coefficients in the Fefferman-Graham expansion of the saddle points along the asymptotic AdS branch of the contour, in terms of the radial AdS coordinate $z = -i\eta$. For example

$$g_{ij}(z) = \frac{1}{z^2} (\bar{\gamma}_{ij} + z^2 \bar{\gamma}_{(2)ij} + \mathcal{O}(z^{3-\sigma})) , \quad (5.11)$$

$$\phi(z) = \bar{\alpha} \bar{\gamma}^{\frac{\lambda_-}{2\sigma}} z^{\lambda_-} - \frac{\bar{\beta}}{\sigma} \bar{\gamma}^{-\frac{\lambda_+}{2\sigma}} z^{\lambda_+} + \mathcal{O}(z^{\lambda_-+1}) . \quad (5.12)$$

³ This means that the leading order asymptotic parameters at this endpoint are related to the NBWF arguments as: $\gamma_{ij} \approx h_{ij}\eta(v)^2$ and $\alpha \approx \chi\eta(v)^{\lambda_-}$.

Hence $\bar{\alpha} \equiv \alpha e^{-i\pi \frac{\lambda_- \lambda_+}{\sigma}}$ and $\bar{\gamma}_{ij} \equiv -\gamma_{ij}$. The minus sign in front of I_{aAdS}^{reg} in (5.10) is connected to the fact that the NBWF behaves as a decaying wave function along the AdS branch of the contour [162]. Euclidean AdS/CFT relates this term to the partition function of a dual field theory. This yields the following holographic form of the semiclassical NBWF in the large volume limit,

$$\Psi[h_{ij}, \chi] = Z_{QFT}^{-1}[\bar{\gamma}_{ij}, \bar{\alpha}] \exp(iS_{ct}[h_{ij}, \chi]/\hbar) . \quad (5.13)$$

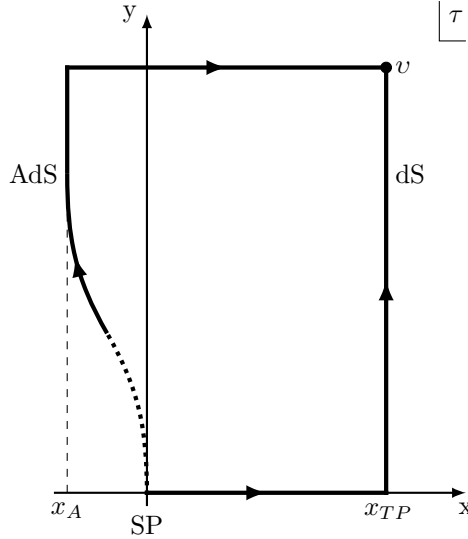


Figure 5.1: Two representations in the complex τ -plane of the same no-boundary saddle point associated with an inflationary universe. Along the vertical part of the AdS contour the geometry is an asymptotically AdS, spherically symmetric domain wall with a complex scalar field profile. Along the vertical branch of the dS contour the saddle point tends to a Lorentzian, inflationary universe. The logarithm of the amplitude of this universe is given by the AdS domain wall action. The horizontal branch of the AdS contour connecting AdS to dS automatically regularizes the AdS action.

5.2.4 Classically from a holographic viewpoint?

Equation (5.13) shows that to leading order in \hbar the probabilities of all asymptotically locally dS histories in the no-boundary state are given by the inverse of the partition function of a Euclidean AdS/CFT dual field theory defined on the boundary surface Σ . The arguments (h_{ij}, χ) of the wave function

enter as external sources $(\bar{\gamma}_{ij}, \bar{\alpha})$ in Z_{QFT} . The dependence of the partition function on the values of the sources therefore gives a holographic measure on asymptotically locally de Sitter configurations.

The probability of each individual history is conserved under scale factor evolution as a consequence of the Wheeler-DeWitt equation. However, the wave function itself oscillates rapidly in the large volume regime. In fact it appears that the large phase factor in (5.13) is crucial in order for the classicality conditions (5.4) to hold and hence for the wave function to predict classical Lorentzian evolution in the first place. This raises the question whether the emergence of classical spacetime evolution can be understood from the dual partition function.

To answer this we will in the next section rewrite the asymptotic wave function in terms of superspace coordinates $(\bar{\gamma}_{ij}, \bar{\alpha})$ that are natural and meaningful from an asymptotic viewpoint and in particular enter as sources in Z . Since this involves a map between two symplectic manifolds we will work in the Hamiltonian formulation of the theory, where the symplectic structure is manifest. The relation between the two wave functions resembles and generalizes the Fourier transformation between a quantum mechanical wave function in momentum and position space. In Section 5.4 we then revisit the classicality conditions in terms of the new superspace coordinates.

We note that a calculation in the same spirit was done in the context of AdS holography in [125], where variables on phase space were identified that make the variational problem well-defined. These variables were found to be related by a canonical transformation to the original AdS fields and momenta. It was established that this transformation is equivalent to holographic renormalization. Furthermore, the generating function of this canonical transformation coincides exactly with the local boundary terms that regularize the AdS action. In the next section we show that this conclusion carries over to the dS case where the local boundary terms appear to play a physical role as discussed above.

5.3 Asymptotic Wave Function

We now derive the NBWF as a function of a new set of variables that remain finite and physically meaningful in the large volume limit. These variables come in canonically conjugate pairs so they are related to the original fields and momenta through a canonical transformation.

5.3.1 The Hamiltonian NBWF

We work with the ADM form of the saddle point metrics (5.7) and with the NBWF in Hamiltonian form⁴,

$$\Psi(h_{ij}, \chi) \propto \int \mathcal{D}g_{ij} \mathcal{D}\phi \mathcal{D}\pi_{(g)}^{ij} \mathcal{D}\pi_{(\phi)} \mathcal{D}N \mathcal{D}N^i \cdot e^{\frac{1}{\hbar} \int d\lambda \int d^3x \left[i\pi_{(\phi)} \dot{\phi} + i\pi_{(g)}^{ij} \dot{g}_{ij} - \sqrt{g} (NH + N^i H_i) \right]}, \quad (5.14)$$

where $\pi_{(\phi)}$ and $\pi_{(g)}^{ij}$ are the conjugate momenta of the scalar field and the metric, a dot means a derivative with respect to λ and where we introduced

$$\begin{aligned} H &= \frac{2\kappa}{g} \left(\pi_{(g)}^{ij} \pi_{(g)ij} - \frac{\text{Tr}(\pi_{(g)})^2}{2} \right) + \frac{(\pi_{(\phi)})^2}{2g} \\ &\quad + \frac{1}{2\kappa} \left(-^{(3)}R + 2\Lambda \right) + \frac{1}{2} g^{ij} \partial_i \phi \partial_j \phi + V(\phi), \\ H_i &= -2i \mathcal{D}^j \left(\frac{\pi_{(g)ij}}{\sqrt{g}} \right) + i \frac{\pi_{(\phi)}}{\sqrt{g}} \partial_j \phi, \end{aligned} \quad (5.15)$$

with \mathcal{D}^j the covariant derivative on slices of constant λ . Also, $^{(3)}R$ is the three dimensional Ricci scalar constructed from g_{ij} . Performing the Gaussian integrations over $\pi_{(\phi)}$ and $\pi_{(g)}^{ij}$ by substituting their extremizing values yields the original action (5.2).

Variation of the action with respect to N and N_i yields the first-class Hamiltonian and momentum constraints. One can use the gauge freedom of coordinate reparameterizations to fix the values of these fields, e.g. $N^i = 0$ and $N = 1$. Concerning the lapse, a change $N(\lambda) \rightarrow N(\lambda) + f'(\lambda)$ can be absorbed in a redefinition of the time coordinate λ to keep the metric invariant. However, the constraint that the range of λ remains unchanged means $v \equiv \int N d\lambda$ is invariant under gauge transformations. To differentiate between the physical data v and all other – gauge dependent – information in N , we introduce the following variables

$$\tau(\lambda) = \int_0^\lambda d\lambda' N(\lambda'), \quad \int \mathcal{D}N(\lambda) \rightarrow \int \mathcal{D}\tau(\lambda). \quad (5.16)$$

The only physical information contained in $\tau(\lambda)$ is $\tau(1) = v$. Therefore, the path integral over N reduces to a physically irrelevant constant of proportionality –

⁴ To simplify notation we set the Hubble constant $H = 1$ in this section.

the volume of the gauge group – and an ordinary integral $\int dv$ over all possible values of $\int d\lambda N(\lambda)$.⁵

The asymptotic momenta in the Hamiltonian version of the action can be expressed in terms of the coefficients in the Fefferman-Graham (FG) expansion (5.9) as follows:

$$\begin{aligned}\pi_{(\phi)} &= i\sqrt{g}(\phi' - N^i \partial_i \phi) \\ &= -\lambda_- \alpha \gamma^{\frac{\lambda_+}{2\sigma}} \eta^{-\lambda_+} + \lambda_+ \frac{\beta}{\sigma} \gamma^{-\frac{\lambda_-}{2\sigma}} \eta^{-\lambda_-} + \mathcal{O}(\eta^{-\lambda_++1}) ,\end{aligned}\quad (5.20)$$

$$\begin{aligned}\pi_{(g)}^i{}_j &= \pi_{(g)}^{ik} g_{kj} = i \frac{\sqrt{g}}{2\kappa} (K^{ik} - K g^{ik}) g_{kj} \\ &= \frac{\sqrt{g}}{2\kappa} \left[-2\delta_j^i - \eta^2 \left(\gamma_{(2)j}^i - \gamma_{(2)} \delta_j^i \right) - \lambda_- \eta^{3-\sigma} \left(\gamma_{(-)j}^i - \gamma_{(-)} \delta_j^i \right) \right. \\ &\quad \left. - \frac{3}{2} \eta^3 \left(\gamma_{(3)j}^i - \gamma_{(3)} \delta_j^i \right) - \lambda_+ \eta^{3+\sigma} \left(\gamma_{(+)j}^i - \gamma_{(+)} \delta_j^i \right) + \mathcal{O}(\eta^4) \right] ,\end{aligned}\quad (5.21)$$

where prime denotes a derivative with respect to τ . These relations follow from the on-shell expression of the momenta in terms of time derivatives of the fields and will be useful below to motivate the change of coordinates we apply there.

Note that the equations of motion of the momenta provide an alternative way to determine the coefficients in the FG expansion order by order, with γ_{ij} , $\gamma_{(3)ij}$, α and β as the only independent coefficients [163]. For instance, the equation for $\pi_{(\phi)}$ implies $\lambda_{\pm} = 3/2 \left(1 \pm \sqrt{1 - 4m^2/9} \right)$. Similarly, the leading-

⁵ This can also be seen from an explicit discretization of the measure. Writing

$$\int \mathcal{D}N = \lim_{J \rightarrow \infty} \int \left(\prod_{m=0}^J dN_m \right) , \quad (5.17)$$

where the subscript m labels the λ -slice, one can consider the following change of variables:

$$M_m \equiv \sum_{n=0}^m N_n , \quad \prod_{m=0}^J dN_m = \prod_{m=0}^J dM_m . \quad (5.18)$$

The second equation follows from the fact that the Jacobian is 1, since the transformation matrix is triangular with only 1's on the diagonal. One can now easily separate the physical quantity $v = M_J$ from the rest. Hence for $m < J$, M_m has no physical significance and we can gauge it away. Therefore,

$$\int \mathcal{D}N = \lim_{J \rightarrow \infty} \int \left(\prod_{m=0}^{J-1} dM_m \right) dM_J = \lim_{J \rightarrow \infty} \int \left(\prod_{m=0}^{J-1} dM_m \right) dv \propto \int dv . \quad (5.19)$$

order equation for $\pi_{(g)}^{ij}$ is satisfied if $\Lambda = 3$ (remember that we are working in units of the de Sitter length) and the next orders imply

$$\begin{aligned}\gamma_{(-)ij} &= -\frac{\kappa}{4}\gamma^{\frac{\lambda_-}{\sigma}}\alpha^2\gamma_{ij} , \\ \gamma_{(2)ij} &= \left(R_{(\gamma)ij} - \frac{1}{4}R_{(\gamma)}\gamma_{ij}\right) , \\ \gamma_{(+)ij} &= -\frac{\kappa}{4}\gamma^{-\frac{\lambda_+}{\sigma}}\frac{\beta^2}{\sigma^2}\gamma_{ij} ,\end{aligned}\tag{5.22}$$

where $R_{(\gamma)ij}$ and $R_{(\gamma)}$ are the Ricci tensor and scalar constructed from γ_{ij} . Finally, defining $\pi_{(\gamma)}^{ij}$ as the coefficient of the $\mathcal{O}(\eta^3)$ -term in (5.22), i.e. $\pi_{(\gamma)}^{ij} \equiv \frac{3\sqrt{\gamma}}{4\kappa}(\gamma_{(3)}\gamma^{ij} - \gamma_{(3)}^{ij})$, we note that the Hamiltonian and momentum constraints require that

$$\text{Tr } \pi_{(\gamma)} = \frac{\lambda_+\lambda_-\alpha\beta}{\sigma} , \quad \mathcal{D}_{(0)}^j \pi_{(\gamma)ij} = \frac{1}{2\sigma}(\lambda_-\alpha\partial_i\beta + \lambda_+\beta\partial_i\alpha) . \tag{5.23}$$

5.3.2 Canonical transformation to asymptotic coordinates

We now proceed by describing the canonical transformation to variables that are meaningful asymptotically. Inspired by the FG expansions (5.9), we introduce a new set of coordinates and conjugate momenta $(\Gamma_{ij}, \Pi_{(\Gamma)}^{ij}, A, B, \eta)$ on extended phase space as follows:

$$\begin{aligned}g_{ij} &\equiv \frac{1}{\eta^2} \left[\Gamma_{ij} + \eta^2 \left(R_{(\Gamma)ij} - \frac{1}{4}R_{(\Gamma)}\Gamma_{ij} \right) - \frac{\kappa}{4}\eta^{3-\sigma}A^2\Gamma^{\frac{\lambda_-}{\sigma}}\Gamma_{ij} \right. \\ &\quad \left. + \frac{2\kappa}{3\sqrt{\Gamma}}\eta^3(\Pi_{(\Gamma)}\Gamma_{ij} - 2\Pi_{(\Gamma)ij}) - \frac{\kappa}{4\sigma^2}\eta^{3+\sigma}B^2\Gamma^{-\frac{\lambda_+}{\sigma}} \right] , \\ \pi_{(g)}^{ij} &\equiv \frac{1}{2\kappa}\frac{\sqrt{\Gamma}}{\eta} \left[-2\Gamma^{ij} + \eta^2 \left(R_{(\Gamma)}^{ij} - \frac{1}{4}R_{(\Gamma)}\Gamma^{ij} \right) \right] + \frac{\sigma-2}{8}\eta^{2-\sigma}A^2\Gamma^{\frac{3}{2\sigma}}\Gamma^{ij} \\ &\quad + \frac{\eta^2}{3} \left(\Pi_{(\Gamma)}\Gamma^{ij} - \Pi_{(\Gamma)}^{ij} \right) - \frac{\sigma+2}{8\sigma^2}\eta^{2+\sigma}B^2\Gamma^{-\frac{3}{2\sigma}}\Gamma^{ij} , \\ \phi &\equiv A\Gamma^{\frac{\lambda_-}{2\sigma}}\eta^{\lambda_-} - \frac{B}{\sigma}\Gamma^{-\frac{\lambda_+}{2\sigma}}\eta^{\lambda_+} , \\ \pi_{(\phi)} &\equiv -\lambda_-A\Gamma^{\frac{\lambda_+}{2\sigma}}\eta^{-\lambda_+} + \frac{\lambda_+}{\sigma}B\Gamma^{-\frac{\lambda_-}{2\sigma}}\eta^{-\lambda_-} ,\end{aligned}\tag{5.24}$$

where $R_{(\Gamma)}$ is the Ricci scalar associated with the metric Γ_{ij} and Γ is its determinant. We raise and lower indices of the new phase space coordinates by acting with Γ_{ij} and its inverse Γ^{ij} .

Note that $(\Gamma_{ij}, \Pi_{(\Gamma)}^{ij}, A, B)$ are functions of the scale factor variable η , but $\Gamma_{ij} \rightarrow \gamma_{ij}$, $\Pi_{(\Gamma)}^{ij} \rightarrow \pi_{(\gamma)}^{ij}$, $A \rightarrow \alpha$ and $B \rightarrow \beta$ when $\eta \rightarrow 0$. The η -dependence merely incorporates the higher order terms in the FG expansions⁶.

Since we only consider compact three-geometries, we can fix the scale of η in (5.8) by choosing $\eta_0 = 1/\sqrt{\text{Vol}(h_{ij})}$. This guarantees that Γ_{ij} has volume 1 at late times.

What we would like to find, is a generating function f , such that

$$\pi_{(g)}^{ij} dg_{ij} + \pi_{(\phi)} d\phi + \sqrt{g} H \frac{d\eta}{\eta} = \Pi_{(\Gamma)}^{ij} d\Gamma_{ij} + B dA + \sqrt{\Gamma} \tilde{H} \frac{d\eta}{\eta} + df + \mathcal{P}(\eta) , \quad (5.25)$$

where $\mathcal{P}(\eta)$ is a function consisting of dA , dB , $d\Gamma$ and $d\Pi_{(\Gamma)}$ terms that cannot be written in the form of one of any of the other terms in (5.25). We will find that $\mathcal{P}(\eta)$ is higher order in η , as expected. Using the definition of the dynamical asymptotic variables (5.24), the left-hand side of (5.25) can be expressed in terms of Γ_{ij} , $\Pi_{(\Gamma)}^{ij}$, A and B . For the scalar field variables we get

$$\begin{aligned} \pi_{(\phi)} d\phi = B dA - d \left(\lambda_- \Gamma^{\frac{3}{2\sigma}} \eta^{-\sigma} \frac{A^2}{2} - \frac{\lambda_-}{\sigma} AB + \frac{\lambda_+}{\sigma^2} \Gamma^{-\frac{3}{2\sigma}} \eta^\sigma \frac{B^2}{2} \right) \\ + \left(\frac{\lambda_-}{4} \eta^{-\sigma} A^2 \Gamma^{\frac{3}{2\sigma}} + \frac{\lambda_+}{4\sigma^2} \eta^\sigma B^2 \Gamma^{-\frac{3}{2\sigma}} \right) \Gamma^{ij} d\Gamma_{ij} \\ - \left(\frac{3\lambda_-}{2} A^2 \Gamma^{\frac{3}{2\sigma}} \eta^{-\sigma} - 2 \frac{\lambda_+ \lambda_-}{\sigma} AB + \frac{3\lambda_+}{2} \frac{B^2}{\sigma^2} \Gamma^{-\frac{3}{2\sigma}} \eta^\sigma \right) \frac{d\eta}{\eta} , \end{aligned} \quad (5.26)$$

where we have used that \sqrt{g} can be expanded as $\sqrt{\Gamma}/\eta^3$ and terms of higher order in η ,

$$\begin{aligned} \sqrt{g} = \frac{\sqrt{\Gamma}}{\eta^3} \left(1 + \frac{\eta^2 R_{(\Gamma)}}{8} - \frac{\eta^{3-\sigma} 3\kappa \Gamma^{\frac{\lambda_-}{\sigma}} A^2}{2} \right. \\ \left. + \frac{\eta^3 \kappa \Pi_{(\Gamma)}}{3\sqrt{\Gamma}} - \frac{\eta^{3+\sigma} 3\kappa \Gamma^{-\frac{\lambda_+}{\sigma}}}{8\sigma^2} + \mathcal{O}(\eta^4) \right) . \end{aligned} \quad (5.27)$$

⁶ The η -dependence of the new variables guarantees that (5.24) holds exactly at all times. Specifically, A contains all terms that are higher order in η , such as the terms with coefficient α_1 in (5.9) and higher order terms of the form $\eta^{\lambda_- + n}$ with $n > 0$. Similarly the corrections to B are of the form $\eta^{\lambda_+ + n}$. The same goes for Γ_{ij} .

Similarly, the gravitational part of the symplectic form is

$$\begin{aligned}
& \pi_{(g)}^{ij} dg_{ij} \\
&= \mathcal{P}(\eta) - \frac{2}{\kappa} d\sqrt{g} + \left[\frac{1}{2\kappa} \frac{\sqrt{\Gamma}}{\eta} \left(\frac{1}{2} R_{(\Gamma)} \Gamma^{ij} - R_{(\Gamma)}^{ij} \right) \right. \\
&\quad \left. - \frac{\lambda_-}{4} \eta^{-\sigma} A^2 \Gamma^{\frac{3}{2\sigma}} \Gamma^{ij} + \Pi_{(\Gamma)}^{ij} - \frac{\lambda_+}{4\sigma^2} \eta^\sigma B^2 \Gamma^{-\frac{3}{2\sigma}} \Gamma^{ij} \right] d\Gamma_{ij} \\
&\quad + \left(-\frac{1}{2\kappa} \frac{\sqrt{\Gamma}}{\eta} R_{(\Gamma)} + \frac{3\lambda_-}{2} \eta^{-\sigma} A^2 \Gamma^{\frac{3}{2\sigma}} - 2\Pi_{(\Gamma)} + \frac{3\lambda_+}{2\sigma^2} \eta^\sigma B^2 \Gamma^{-\frac{3}{2\sigma}} \right) \frac{d\eta}{\eta} ,
\end{aligned} \tag{5.28}$$

where \sqrt{g} is given in (5.27). Adding these expressions one sees that the terms proportional to $A^2 d\Gamma_{ij}$ and $B^2 d\Gamma_{ij}$ cancel. Furthermore, the term proportional to $d\Gamma_{ij}/\eta$ in (5.29) is a total derivative,

$$\frac{1}{2\kappa} \frac{\sqrt{\Gamma}}{\eta} \left[\left(\frac{1}{2} R_{(\Gamma)} \Gamma^{ij} - R_{(\Gamma)}^{ij} \right) d\Gamma_{ij} - R_{(\Gamma)} \frac{d\eta}{\eta} \right] = d \left(\frac{1}{2\kappa} \frac{\sqrt{\Gamma}}{\eta} R_{(\Gamma)} \right) . \tag{5.29}$$

Finally, expanding \sqrt{g} in the first term of (5.29) using (5.27), one obtains (5.25) with

$$\begin{aligned}
f &= \frac{\sqrt{\Gamma}}{\kappa} \left(\frac{-2}{\eta^3} + \frac{1}{4\eta} R_{(\Gamma)} \right) + \frac{\sigma}{4} \eta^{-\sigma} \Gamma^{\frac{3}{2\sigma}} A^2 \\
&\quad - \frac{2}{3} \Pi_{(\Gamma)} + \frac{\lambda_-}{\sigma} AB - \frac{1}{4\sigma} \Gamma^{-\frac{3}{2\sigma}} \eta^\sigma B^2 ,
\end{aligned} \tag{5.30}$$

and

$$\sqrt{\Gamma} \tilde{H} \frac{d\eta}{\eta} = \sqrt{g} H \frac{d\eta}{\eta} + 2 \left(\frac{\lambda_+ \lambda_-}{\sigma} AB - \Pi_{(\Gamma)} \right) \frac{d\eta}{\eta} . \tag{5.31}$$

Furthermore, $\mathcal{P}(\eta)$ indeed only contains terms that are higher order in η . Note that the additional term in the new Hamiltonian \tilde{H} vanishes exactly when the Hamiltonian constraint is applied. This means that the physical constraint remains the same, as it should.

5.3.3 A new wave function $\tilde{\Psi}$

We now implement the asymptotically canonical transformation to the variables $(\Gamma_{ij}, \Pi_{(\Gamma)}^{ij}, A, B, \eta)$ at the level of the wave function and write the NBWF in

terms of these new variables. In the $\eta \rightarrow 0$ limit, this reduces to an asymptotic wave function $\tilde{\Psi}_{as}$ which is a function of (γ_{ij}, α) .

The exponent in the Hamiltonian version (5.14) of the path-integral has the same structure as the symplectic one-form in (5.25), with differentials replaced by time derivatives, and without the $\mathcal{P}(\eta)$ term in (5.25). Including this as a correction to the new Hamiltonian \tilde{H} which vanishes in the $\eta \rightarrow 0$ limit, we can use (5.25) to rewrite the exponent in (5.14) in terms of the new variables. Furthermore, since the Jacobian of a canonical transformation equals one, the measure is left invariant⁷. Hence we can write, for $\eta_* = \eta(v)$,

$$\begin{aligned} \Psi(h_{ij}, \chi) \propto \int_{\mathcal{C}} \mathcal{D}\Gamma_{ij} \mathcal{D}\Pi_{(\Gamma)}^{ij} \mathcal{D}A \mathcal{D}B d\log(\eta_*) e^{\frac{i}{\hbar} \int d\eta \int d^3x \frac{d\tilde{f}}{d\eta}(\Gamma_{ij}, \Pi_{(\Gamma)}^{ij}, A, B, \eta)} \\ \cdot e^{+\frac{i}{\hbar} \int d\eta \int d^3x \left(B \frac{dA}{d\eta} + \Pi_{(\Gamma)}^{ij} \frac{d\Gamma_{ij}}{d\eta} + \sqrt{\Gamma} \frac{\tilde{H}}{\eta} \right)}, \end{aligned} \quad (5.32)$$

where the integral is still over the class \mathcal{C} of histories that obey the no-boundary conditions of regularity and compactness and that match (h_{ij}, χ) on the boundary Σ . That is,

$$g_{ij}(\eta_*, \Gamma_{ij}, \Pi_{(\Gamma)}^{ij}, A, B) = h_{ij}, \quad \phi(\eta_*, \Gamma_{ij}, \Pi_{(\Gamma)}^{ij}, A, B) = \chi. \quad (5.33)$$

Solving the conditions (5.33) yields an expression of the boundary values of the momenta in terms of the old coordinates, the new coordinates and η_* ,

$$\Pi_{(\Gamma)}^{ij}(\Gamma_{ij}, A, h_{ij}, \chi, \eta_*) , \quad B(\Gamma_{ij}, A, h_{ij}, \chi, \eta_*) . \quad (5.34)$$

The term with the generating function is a boundary term at $\eta = \eta_*$. Substituting the solutions (5.34) in f defines a new function \tilde{f} that does not depend on the momenta but is a function of the coordinates only – both the original ones and the new asymptotic coordinates,

$$\begin{aligned} \tilde{f} = \frac{\sqrt{\Gamma}}{\kappa \eta_*^3} \left[1 + \eta_*^2 \left(\Gamma^{ij} h_{ij} + \frac{R_{(\Gamma)}}{2} \right) \right] \\ - \frac{\sigma}{2} \eta_*^{-\sigma} A^2 \Gamma^{\frac{3}{2\sigma}} + \sigma \eta_*^{-\lambda_+} A \chi \Gamma^{\frac{\lambda_+}{2\sigma}} - \frac{\lambda_+}{2} \frac{\sqrt{\Gamma}}{\eta_*^3} \chi^2 . \end{aligned} \quad (5.35)$$

Finally, by inserting the identities

$$\int \mathcal{D}A = \int dA_* \int_{A(\eta_*)=A_*} \mathcal{D}A , \quad \int \mathcal{D}\Gamma_{ij} = \int d\Gamma_{*ij} \int_{\Gamma_{ij}(\eta_*)=\Gamma_{*ij}} \mathcal{D}\Gamma_{ij} , \quad (5.36)$$

⁷ This again only holds up to $\mathcal{O}(\eta)$. We will not consider those correction terms because they higher order in \hbar .

in the wave function we can take the generating function \tilde{f} outside the path integral, because it depends only on the (fixed) boundary values. This yields

$$\Psi(h_{ij}, \chi) = \int d\Gamma_{*ij} dA_* d\log(\eta_*) e^{\frac{i}{\hbar} \int d^3x \tilde{f}(\Gamma_{*ij}, A_*, h_{ij}, \chi, \eta_*)} \tilde{\Psi}(\Gamma_{*ij}, A_*, \eta_*) , \quad (5.37)$$

with the no-boundary wave function $\tilde{\Psi}(\Gamma_{*ij}, A_*, \eta_*)$ in terms of the asymptotic variables given by

$$\tilde{\Psi}(\Gamma_{*ij}, A_*, \eta_*) \equiv \int_C \mathcal{D}\Gamma_{ij} \mathcal{D}\Pi_{(\Gamma)}^{kl} \mathcal{D}A \mathcal{D}B e^{\frac{i}{\hbar} \int d\eta \int d^3x \left(B \frac{dA}{d\eta} + \Pi_{(\Gamma)}^{ij} \frac{d\Gamma_{ij}}{d\eta} + \sqrt{\Gamma} \frac{\tilde{H}}{\eta} \right)} . \quad (5.38)$$

Since the transformation to new coordinates is canonical for $\eta \rightarrow 0$, the structure of the path integral representing the new wave function $\tilde{\Psi}$ is similar to that of the original wave function, only with the new Hamiltonian \tilde{H} replacing the original H .

The relation (5.37) can be inverted by considering (5.25) in the other direction, with the generating function f subtracted from both sides. A similar derivation, starting by inverting the relations (5.24), then yields

$$\tilde{\Psi}(\Gamma_{*ij}, A_*, \eta_*) = \int dh_{ij} d\chi e^{-\frac{i}{\hbar} \int d^3x \tilde{f}(\Gamma_{*ij}, A_*, h_{ij}, \chi, \eta_*)} \Psi(h_{ij}, \chi) . \quad (5.39)$$

The derivation of (5.37) together with the definition of $\tilde{\Psi}$ is the central result of this paper. The new wave function is obtained from the original wave function by a transformation that generalizes the Fourier transform. The generating function f is a finite polynomial, given in (5.30). Furthermore, since the dynamical η -dependent asymptotic variables tend to constants at late times, $\tilde{\Psi}$ converges to the asymptotic wave function $\tilde{\Psi}_{as}(\gamma_{ij}, \alpha)$ where (γ_{ij}, α) do not depend on η . This is the form of the wave function of the universe that is directly related to and potentially computed with dS/CFT techniques.

To gain further intuition about the relation between the two formulations of the wave function we consider the semiclassical approximation of both Ψ and $\tilde{\Psi}$. In the semiclassical approximation $\Psi(h_{ij}, \chi)$ can be written as

$$\Psi(h_{ij}, \chi) \propto e^{-\frac{1}{\hbar} I_{\text{extr}}[h_{ij}, \chi]} , \quad (5.40)$$

where I_{extr} is the action of a regular compact saddle point, i.e. an extremizing solution to the equations of motion, that satisfies the boundary condition that $g_{ij} = h_{ij}$ and $\phi = \chi$ at the boundary Σ where $\eta = \eta_*$. For simplicity we assume here there is only one such saddle point.

Similarly, we substitute the semiclassical form of $\tilde{\Psi}(\Gamma_{*ij}, A_*, \eta_*)$ in (5.37),

$$\int d\Gamma_{*ij} dA_* d\log(\eta_*) e^{\frac{i}{\hbar} \int d^3x \tilde{f}(\Gamma_{*ij}, A_*, h_{ij}, \chi, \eta_*)} e^{-\frac{1}{\hbar} \tilde{I}_{\text{extr}}[\Gamma_{*ij}, A_*, \eta_*]} . \quad (5.41)$$

Solving the remaining integral in the steepest descent approximation yields the following relations,

$$i \int d^3x \frac{\partial \tilde{f}}{\partial \Gamma_{*ij}} = \frac{\partial \tilde{I}}{\partial \Gamma_{*ij}} , \quad i \int d^3x \frac{\partial \tilde{f}}{\partial A_*} = \frac{\partial \tilde{I}}{\partial A_*} , \quad i \int d^3x \frac{\partial \tilde{f}}{\partial \eta_*} = \frac{\partial \tilde{I}}{\partial \eta_*} . \quad (5.42)$$

We denote the solutions of these equations by Γ_{*ij} , A_* and η_* . They are functions of the original data, h_{ij} and χ . The semiclassical approximation thus gives the following relation between the two extremizing actions,

$$I_{\text{extr}}[h_{ij}, \chi] = \tilde{I}_{\text{extr}}[\Gamma_{*ij}, A_*, \eta_*] - i \int d^3x \tilde{f}(\Gamma_{*ij}, A_*, h_{ij}, \chi, \eta_*) . \quad (5.43)$$

It is important to notice that in the above equation h_{ij} , χ , Γ_{*ij} , A_* and η_* are related to each other by (5.42), no new variables have been introduced here⁸. One can use this to determine the behavior of the on-shell actions in the large volume limit. The asymptotic behavior of the action $I_{\text{extr}}(h_{ij}, \chi)$ in terms of the asymptotic expansions given in (5.9) is known to be

$$I_{\text{extr}} \approx \frac{i}{\kappa} \int d^3x \sqrt{\gamma} \left(\frac{2}{\eta_*^3} - \frac{R(\gamma)}{4\eta_*} \right) - \frac{\kappa}{4\sigma} \left[\alpha^2 \sigma^2 \gamma^{\frac{\lambda_-}{\sigma}} \eta_*^{-\sigma} - \beta^2 \gamma^{-\frac{\lambda_+}{\sigma}} \eta_*^{\sigma} \right] + \mathcal{O}(\eta_*) + I_{\text{IR}} , \quad (5.44)$$

where I_{IR} is an η_* -independent constant that depends on the non-asymptotic behavior of the on-shell action, i.e. on the value of the fields around the SP. Hence the diverging terms in the on-shell action I are equal to those of the generating function (5.30). This means there are no diverging terms left in \tilde{I} : the on-shell action is regulated by the canonical transformation to asymptotically meaningful coordinates, as expected⁹. This is of course consistent with the usual counterterms employed in dS/CFT, which are given by

$$S_{\text{ct}}[h, \chi] = -\frac{2i}{\kappa} \int d^3x \sqrt{h} + \frac{i}{2\kappa} \int d^3x \sqrt{h} {}^{(3)}R + \frac{i\lambda_-}{2} \int d^3x \sqrt{h} \chi^2 + \dots , \quad (5.45)$$

⁸ The relations between these variables is in leading order $h_{ij} \approx \Gamma_{*ij} \eta_*^{-2}$ and $\chi \approx A_* \eta_*^{\lambda_-}$

⁹ This can also be seen by writing $\tilde{I}_{\text{extr}} = -i \int (\Pi_{(\Gamma)}^{ij} d\Gamma_{ij} + B dA + \sqrt{\Gamma} \tilde{H} d\eta/\eta)$. The Hamiltonian \tilde{H} vanishes on-shell, as a result of the Hamiltonian constraint, and the other terms in the on-shell action remain finite, by virtue of the finiteness of the asymptotic variables. Therefore, \tilde{I}_{extr} cannot diverge.

where the dots refer to additional scalar counterterms that enter for certain scalar masses only. The divergent parts of the counterterms S_{ct} are equal to those of f .

5.3.4 Implications for dS/CFT

The relation between the semiclassical actions in (5.43) is reminiscent of the relation (5.10) between the action computed in the dS and in the AdS representations of the NBWF saddle points and leads, in the large volume limit, to the following chain of equalities,

$$\tilde{I}_{\text{extr}}[\Gamma_{*ij}, A_*, \eta_*] = I_{\text{extr}}[h_{ij}, \chi] - iS_{ct} = -I_{a\text{AdS}}^{\text{reg}}[\bar{\gamma}_{*ij}, \bar{\alpha}_*], \quad (5.46)$$

where $\bar{\gamma}_{ij}$ and $\bar{\alpha}$ are the natural variables from an AdS viewpoint, defined below (5.10), and thus *locally related* to the argument of the wave function. Using (5.13) this leads to the following formulation of a semiclassical dS/CFT correspondence

$$\tilde{\Psi}_{as}(\gamma_{ij}, \alpha) = \frac{1}{Z_{\text{QFT}}(\bar{\gamma}_{ij}, \bar{\alpha})}, \quad (5.47)$$

where we remind the reader that Z_{QFT} is the partition function of a deformation of a Euclidean AdS/CFT dual. This is both a more elegant and a cleaner formulation of dS/CFT than (5.13), since it is stated purely in terms of quantities that are available in the dual QFT. There is no need to involve the local counterterms in the bulk. Evaluating the wave function at finite scale factor rather than asymptotically loosely corresponds to adding a UV regulator v in the boundary theory. In what follows we fix the overall scale of the boundary metric to have volume one. This is consistent with $\text{Vol}(\Gamma_{ij}) \rightarrow 1$, as implied by our choice of η_0 above.

The variables with a bar differ from the original variables by a phase only. More specifically, $\bar{\gamma}_{ij}$ and $\bar{\alpha}$ are the coefficients that are real in the FG expansion in the asymptotic AdS domain. Whereas η is real on the dS part of the contour, the radial AdS variable $z = -i\eta$. Using the analyticity of (5.9), one can write the FG expansions in terms of z and the barred variables, with

$$\bar{\Gamma}_{ij} = -\Gamma_{ij}, \quad \bar{\Pi}_{(\Gamma)}^{ij} = \Pi_{(\Gamma)}^{ij}, \quad \bar{A} = e^{-i\pi \frac{\lambda_- \lambda_+}{\sigma}} A, \quad \bar{B} = e^{i\pi \frac{\lambda_- \lambda_+}{\sigma}} B. \quad (5.48)$$

The expectation value of the CFT stress tensor is dual to the subleading fall-off of the metric and the bulk scalar field corresponds to the expectation value of

the dual scalar operator,

$$\langle \mathcal{T}^{ij}(\vec{x}) \rangle = \frac{\delta}{\delta \bar{\gamma}_{ij}} \log Z_{\text{QFT}} \approx \frac{\delta \tilde{I}_{\text{extr}}[\Gamma_{*ij}, A_*, \eta_*]}{\delta \Gamma_{*ij}} \rightarrow \bar{\pi}_{(\gamma)}^{ij}(\vec{x}) , \quad (5.49a)$$

$$\langle \mathcal{O}(\vec{x}) \rangle = \frac{\delta}{\delta \bar{\alpha}} \log Z_{\text{QFT}} \approx -e^{i\pi \frac{\lambda - \lambda_+}{\sigma}} \frac{\delta \tilde{I}_{\text{extr}}[\Gamma_{*ij}, A_*, \eta_*]}{\delta A_*} \rightarrow \bar{\beta}(\vec{x}) . \quad (5.49b)$$

Here $\bar{\pi}_{(\gamma)}^{ij}$ and $\bar{\beta}$ are the asymptotic values of $\bar{\Pi}_{(\Gamma)}^{ij}$ and \bar{B} on the extremizing solution for $v \rightarrow i\infty$, i.e. $\eta_* \rightarrow 0$.

The phase factors in the relations (5.48) between the asymptotic coefficients on the dS and the AdS branches means the vevs in the dual are in general complex when the argument of the dS wave function is real. This sharpens the question whether the emergence of classical spacetime evolution in the large volume limit along the dS branch can be understood from the dual partition function.

5.4 Classicality 2.0

The NBWF in its usual form in terms of the variables (h_{ij}, χ) oscillates very rapidly in the large volume limit. The large phase factor means the classicality conditions (5.4) hold almost automatically, so that the wave function predicts that (h_{ij}, χ) evolves classically. In fact the classical behavior can be understood as a consequence of the Wheeler-DeWitt equation in the large volume regime and therefore applies to any wave function in terms of these variables that satisfies the Hamiltonian constraint.

By contrast the wave function $\tilde{\Psi}$ need not oscillate and has no exponent that diverges in the large volume limit. This leads to the question whether it obeys the classicality conditions (5.4) at large volume. The derivation in [147] of the conditions required for a wave function to predict classical evolution is general and applies also to the wave function in terms of the new asymptotic variables. Asymptotically the classicality conditions in the latter formulation are

$$|\nabla_{A_*} \tilde{I}_R| = |\text{Im} B_*(\vec{x})| \ll |\text{Re} B_*(\vec{x})| = |\nabla_{A_*} \tilde{S}| , \quad (5.50a)$$

$$|\nabla_{\Gamma_{*ij}} \tilde{I}_R| = |\text{Im} \Pi_{(\Gamma)}^{ij}(\vec{x})| \ll |\text{Re} \Pi_{(\Gamma)}^{ij}(\vec{x})| = |\nabla_{\Gamma_{*ij}} \tilde{S}| . \quad (5.50b)$$

These conditions are stronger than the original classicality conditions (5.4), derived from the NBWF in its usual ‘bulk’ form. In particular, the original

conditions do not involve the subleading coefficients in the Fefferman-Graham expansion. From the dual viewpoint they involve the sources only whereas (5.50) are requirements on the vevs.

This means that the ensembles of classical histories predicted by both wave functions may not be identical. We investigate and confirm this in the next section in a minisuperspace approximation in which we can verify the classicality conditions and identify the ensemble explicitly.

This difference does not point towards an inconsistency; certain variables may exhibit classical behavior when others don't. One may ask, however, given two different notions of classicality, which one is more physical? While the original classicality conditions are natural from the point of view of an observer in the bulk it is clear that holography suggests the new set of stronger conditions (5.50) is in fact more accurate and appropriate in quantum gravity.

5.5 Minisuperspace model

In this section we compute the ensemble of classical histories predicted by $\tilde{\Psi}$ in a minisuperspace model consisting of homogeneous and isotropic histories. Adhering to the notation introduced above, the saddle point geometries can be written as

$$ds^2 = d\tau^2 + a^2(\tau)d\Omega_3^2, \quad (5.51)$$

where $d\Omega_3^2$ is the metric on the unit three-sphere. As before we consider gravity coupled to a single scalar field described by the action (5.2) in which we take the potential V to be quadratic. The usual NBWF is therefore a function of the boundary values (b, χ) of the scale factor $a(\tau)$ and scalar field $\phi(\tau)$.

In [147] the usual semiclassical NBWF in this minisuperspace approximation was evaluated by systematically solving for the saddle points with the boundary conditions that $a = b$ and $\phi = \chi$ at the boundary $\tau = v$ and that geometry and field are regular at the SP of the instanton where the scale factor vanishes, say at $\tau = 0$.

The boundary conditions mean the saddle point solutions are generically complex. As discussed in Section 5.2 their geometric representation depends on the choice of contour in the complex τ -plane connecting the SP with the endpoint $\tau = v$. Along a commonly used ‘dS contour’ the analysis of [147] identified a one-parameter set of saddle point geometries consisting of a slightly deformed Euclidean four-sphere that makes a smooth transition through a complex intermediate region, to a Lorentzian inflationary history in which the scalar field

slowly rolls down to the minimum of its potential. Those saddle points were found to obey the usual classicality conditions in the large volume limit, leading [147] to conclude that the usual NBWF in this minisuperspace model describes a one-parameter family of inflationary universes that are asymptotically de Sitter. We now generalize this analysis and compare with the classical predictions of $\tilde{\Psi}$ at large volume.

5.5.1 Minisuperspace wave function $\tilde{\Psi}$

We first construct the minisuperspace wave function $\tilde{\Psi}$. The Fefferman-Graham expansions (5.9) in the minisuperspace approximation reduce to

$$\begin{aligned} a &= \frac{\gamma}{\eta} + \frac{\eta}{4\gamma} - \frac{\kappa\alpha^2}{16\pi^2}\gamma^{\frac{9}{\sigma}-2}\eta^{2-\sigma} + \frac{\kappa\gamma(3)}{36\pi^2\gamma}\eta^2 - \frac{\kappa\beta^2}{16\pi^2\sigma^2}\gamma^{-\frac{9}{\sigma}-2}\eta^{2+\sigma} + \mathcal{O}(\eta^3) , \\ \phi &= \frac{\alpha}{\sqrt{2}\pi}\gamma^{\frac{3\lambda_-}{\sigma}}\eta^{\lambda_-} - \frac{\beta}{\sqrt{2}\pi\sigma}\gamma^{-\frac{3\lambda_+}{\sigma}}\eta^{\lambda_+} + \mathcal{O}(\eta^{\lambda_-+1}) , \end{aligned} \quad (5.52)$$

where we have defined¹⁰ $\gamma_{ij} \equiv \gamma^2\Omega_{ij}$. The momenta conjugate to a and ϕ can be found from $\pi_{(a)} = -12\pi^2 i a a' / \kappa$ and $\pi_{(\phi)} = 2\pi^2 i a^3 \phi'$. All coefficients in the expansions (5.52) are given in terms of $(\alpha, \beta, \gamma, \gamma_{(3)})$ by the equations of motion. As before we define time dependent functions $\Gamma(\eta)$, $\Pi_{(\Gamma)}(\eta)$, $A(\eta)$ and $B(\eta)$ such that¹¹ $\Gamma \rightarrow 1$, $\Pi_{(\Gamma)} \rightarrow \pi_{(\gamma)}$, $A \rightarrow \alpha$ and $B \rightarrow \beta$ for $\eta \rightarrow 0$. This allows us to write the expansions as a finite polynomial. The symplectic form (5.25) becomes

$$\pi_{(a)} da + \pi_{(\phi)} d\phi + H \frac{d\eta}{\eta} = \Pi_{(\Gamma)} d\Gamma + B dA + \tilde{H} \frac{d\eta}{\eta} + df + \mathcal{P}(\eta) , \quad (5.53)$$

where \mathcal{P} contains higher order terms in η and where the new Hamiltonian \tilde{H} and generating function f are given by

$$\tilde{H} = H - \Gamma \Pi_{(\Gamma)} + \frac{2m^2}{\sigma} AB , \quad (5.54)$$

$$f = -\frac{4\pi^2\Gamma^3}{\kappa\eta^3} + \frac{3\pi^2\Gamma}{\kappa\eta} + \frac{\lambda_-}{\sigma} AB - \frac{\Gamma\Pi_{(\Gamma)}}{3} + \frac{\sigma}{4}\Gamma^{\frac{9}{\sigma}}A^2\eta^{-\sigma} - \frac{1}{4\sigma}\Gamma^{-\frac{9}{\sigma}}B^2\eta^{\sigma} ,$$

¹⁰ We use in this section the same notation as in the previous sections for simplicity, but be aware that their meaning is not completely the same. For example γ is not the determinant of γ_{ij} anymore. The same goes for the other variables.

¹¹ γ can be fixed to one by an appropriate choice of η_0 in (5.8). Furthermore, $\pi_{(\gamma)}$ is related to $\gamma_{(3)}$ as before.

and the wave function $\tilde{\Psi}$ in terms of the asymptotic variables can be written as

$$\tilde{\Psi}(\Gamma_*, A_*, \eta_*) = \int_{\mathcal{C}} \mathcal{D}\Gamma \mathcal{D}\Pi_{(\Gamma)} \mathcal{D}A \mathcal{D}B \, e^{i \int B dA + i \int \Pi_{(\Gamma)} d\Gamma + i \int \frac{d\eta}{\eta} \tilde{H}(\Gamma, \Pi_{(\Gamma)}, A, B)} , \quad (5.55)$$

where \tilde{f} is obtained from f by substituting the momenta $(B, \Pi_{(\Gamma)})$ in terms of the coordinates (a, ϕ, A, Γ) , yielding

$$\begin{aligned} \tilde{f} = & \frac{\pi^2 \Gamma_*^3}{\eta_*^3} \left(\frac{8}{\kappa} - \lambda_+ \chi^2 \right) - \frac{12\pi^2 b \Gamma_*^2}{\kappa \eta_*^2} \\ & + \sigma A_* \chi \sqrt{2\pi} \Gamma_*^{\frac{3\lambda_+}{\sigma}} \eta_*^{-\lambda_+} + \frac{6\pi^2 \Gamma_*}{\kappa \eta_*} - \frac{\sigma}{2} A_*^2 \Gamma_*^{\frac{9}{\sigma}} \eta_*^{-\sigma} , \end{aligned} \quad (5.56)$$

and where \tilde{H} now includes the higher order terms in η coming from \mathcal{P} , namely

$$\begin{aligned} \mathcal{P}(\eta) = & -\eta^2 \left(\frac{\Pi_{(\Gamma)}}{12\Gamma^2} + \frac{3}{16}(\sigma-1)A^2\Gamma^{\frac{9}{\sigma}-3}\eta^{-\sigma} - \frac{3}{16\sigma^2}(\sigma+1)B^2\Gamma^{-\frac{9}{\sigma}-3}\eta^\sigma \right) d\Gamma \\ & - \frac{\kappa\Gamma^{-\frac{18}{\sigma}-4}}{1728\pi^2\sigma^3}\eta^{3-2\sigma} \left(-4\Gamma\Pi_{(\Gamma)} - 9(\sigma-1)A^2\Gamma^{\frac{9}{\sigma}}\eta^{-\sigma} + 9(\sigma+1)B^2\Gamma^{-\frac{9}{\sigma}}\eta^\sigma \right) \\ & \times \left(4\sigma\Gamma(\Gamma d\Pi_{(\Gamma)} - \Pi_{(\Gamma)}d\Gamma) + 9A\Gamma^{\frac{9}{\sigma}}\eta^{-\sigma}((2\sigma-9)Ad\Gamma - 2\sigma\Gamma dA) \right. \\ & \left. + \frac{9}{\sigma^2}B\Gamma^{-\frac{9}{\sigma}}\eta^\sigma((2\sigma+9)Bd\Gamma - 2\sigma\Gamma dB) \right) . \end{aligned} \quad (5.57)$$

Finally the relation between the different on-shell actions in the semiclassical approximations becomes

$$I_{\text{extr}}(b, \chi) = \tilde{I}_{\text{extr}}(\Gamma_*, \mathbf{A}_*, \boldsymbol{\eta}_*) - i\tilde{f}(\Gamma_*, \mathbf{A}_*, b, \chi, \boldsymbol{\eta}_*) . \quad (5.58)$$

The minisuperspace approximation allows for an explicit calculation of the actions which shows that all divergences of I_{extr} are indeed contained in \tilde{f} .

5.5.2 Classical Predictions

Our main motivation to evaluate $\tilde{\Psi}$ in the minisuperspace approximation is to explicitly verify the differences between the classical predictions of both

formulations of the NBWF implied by the classicality conditions (5.4) expressed in terms of (h_{ij}, χ) or (5.50) terms of (γ_{ij}, α) .

Applied to the minisuperspace model, the classicality conditions (5.50) in terms of the asymptotic variables require that asymptotically classical universes must have real β and real $\gamma_{(3)}$. The latter condition follows from the former, because the Hamiltonian constraint implies $\Gamma\Pi_{(\Gamma)} = 2m^2 AB/\sigma$. Since Γ_* and A_* are real, $\Pi_{(\Gamma*)}$ has to be real if B_* is real, and vice versa.

To evaluate $\tilde{\Psi}$ we first solve the equations of motion subject to the above (no)-boundary conditions. The saddle points can be viewed as solutions $(a(\tau), \phi(\tau))$ in the complex τ -plane, as discussed in Section 5.2. Following [147] we label different solutions by the absolute value ϕ_0 of the scalar field at the SP which we write as $\phi(\tau = 0) = \phi_0 e^{i\theta}$. It follows from the FG expansions that in saddle point solutions associated with asymptotically classical histories, field and geometry become real along a vertical line somewhere in the τ -plane, $\tau = x_{\text{TP}} + iy$. In [147] it was found that for each value of ϕ_0 , there is a single value of θ as well as a vertical line labeled by x_{TP} along which the solutions become real and Lorentzian, satisfying the classicality conditions.

We have performed a more systematic analysis of the classical predictions which we summarize in Figure 5.2 where we show, for three values of the scalar mass m^2 , the values (x_{TP}, θ) in function of ϕ_0 for which the saddle points become real at large times y . As can be seen, there are in general multiple one-parameter families of solutions. This generalizes the results obtained in [147], where only the solutions depicted in red were found.¹²

The parameters specifying the solutions change continuously with the value of the scalar field mass. The conformally coupled scalar with $m^2 = 2$ is a special value for which the space of solutions has an enhanced symmetry. This can be explained analytically, as described in Appendix B of [2].

With the solutions found above, the asymptotic wave function $\tilde{\Psi}$ can be constructed. It suffices to find the relation between the asymptotic parameters (dual to the sources) and the initial conditions denoted by ϕ_0 , in order to interpret the solutions above as saddle points of $\tilde{\Psi}$. In Figure 5.3 we show α as a function of ϕ_0 for the solutions found in Figure 5.2. This figure shows that the correspondence between ϕ_0 and α is not one-to-one. Instead, it can happen that multiple saddle point solutions contribute to $\tilde{\Psi}(\alpha)$ for a single value of α , even within each of the “branches” identified in Figure 5.2. We return to this point below.

¹² Our result is consistent with observations in dimensions other than four [164] and for different potentials [165].

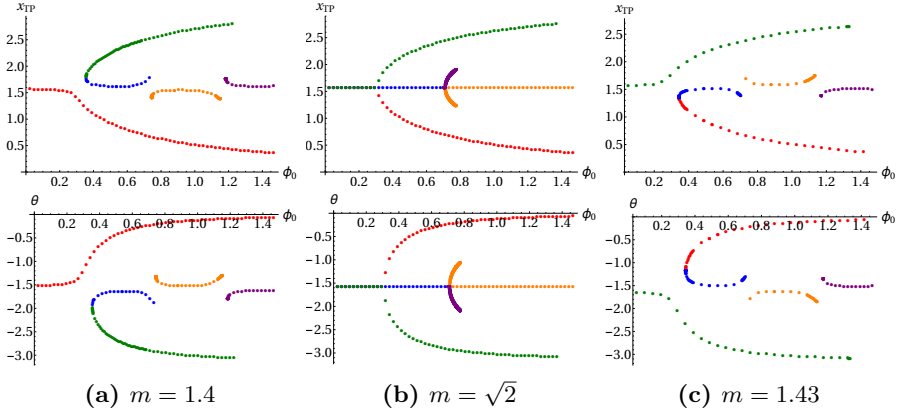


Figure 5.2: Each semiclassical contribution to the NBWF in the minisuperspace approximation can be characterized by three numbers: ϕ_0 , the absolute value of the scalar field at the SP; θ , the phase of the scalar field at the SP; and x_{TP} the real part of τ . This figure shows the values of θ and x_{TP} as a function of ϕ_0 for which both $a(\tau)$ and $\phi(\tau)$ become real and the solutions asymptote to de Sitter space as $y \rightarrow \infty$. The colors indicate different continuously connected sets of solutions that differ only in their relative position in the (x_{tp}, θ) -plane. The solutions in red were previously found in [147]. Depending on the mass of the scalar field, the map of the solutions looks slightly different. We show this for three values of m ; from left to right 1.4, $\sqrt{2}$ and 1.43.

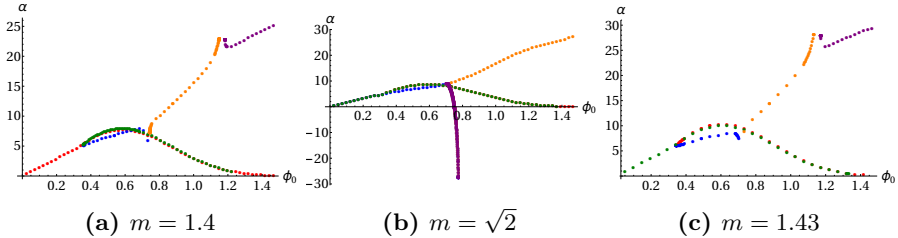


Figure 5.3: The value of α , the coefficient of the leading fall-off of the scalar field profile, versus ϕ_0 , the absolute value of the scalar field on the SP, for the semiclassical contributions to the minisuperspace NBWF shown in Figure 5.2. The color of each branch coincides with the color used there. Notice that the red and green branches almost completely coincide.

We now consider the classicality conditions, starting with their original formulation. Figure 5.4 shows the ratio of the gradients of the real and imaginary parts of the action with respect to the usual variables (b, χ) for the solutions with $m = 1.43$. The other values of the scalar mass give very similar results. If the ratio of the derivatives plotted in Figure 5.4 is small, the usual NBWF

predicts that the corresponding homogeneous, isotropic configuration (b, χ) evolves classically at large volume. As anticipated in Section 5.4, all solutions satisfy these classicality conditions.

Notice that the values in these figures are of the expected order of magnitude. Because the imaginary part of the action goes as $S \sim \eta_*^{-3}$ while I remains of order 1, we expect that $|\nabla I / \nabla S| \sim \eta_*^3$, giving the resulting values of the classicality conditions. However, the semiclassical solutions indicated in red seem to do parametrically better than this. We will see that this distinction between the saddle points carries over to $\tilde{\Psi}$ and appears to be physically meaningful.

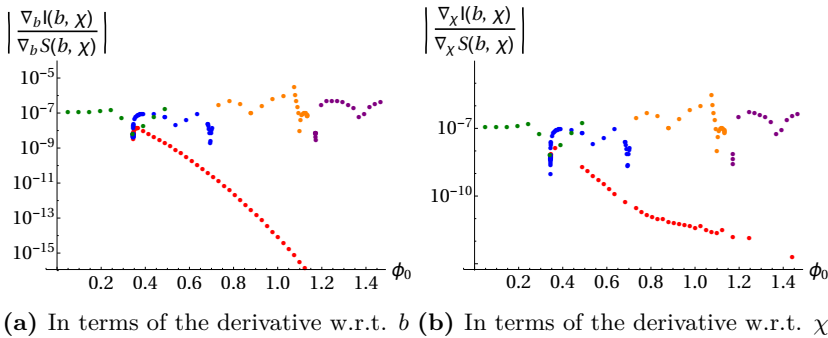


Figure 5.4: The classicality conditions in terms of the variables (b, χ) for the scalar field solutions with $m = 1.43$ given in the rightmost panel of Figure 5.2, evaluated at $\eta_* = e^{-6} \approx 2 \cdot 10^{-3}$. Each contribution is indicated in the same color as in Figure 5.2. As expected, all solutions satisfy (5.4) and are hence predicted to behave classically asymptotically.

Next we evaluate the classicality conditions in terms of asymptotic variables. Figure 5.5 shows the ratio of the imaginary part of β to its real part as a function of ϕ_0 for three different masses of the scalar field¹³. A number of points are important. First, the scale on the y -axis is very different from the scale in Figure 5.4 – the ratio is a lot larger. In fact, for some solutions the ratio is of order 1 or even larger. Hence these solutions do not obey the more stringent classicality conditions in terms of asymptotic variables. More precisely, none of the solutions that were previously unknown are classical. A remarkable observation is that for small ϕ_0 there are no solutions at all that are predicted to behave classically, even though a perturbative analysis based on the original classicality conditions leads to the opposite conclusion [147]. Note also that for the conformal mass, the red and green branch coincide again and correspond to classical solutions. This was to be expected due to the symmetry of this case

¹³ From the holographic point of view, it would be more natural to plot these as functions of α , but in this way it is easier to compare these results with the classicality conditions in bulk variables.

(see Appendix B of [2]). It is interesting to observe that a small breaking of this symmetry – for example changing the mass from $\sqrt{2}$ to 1.4 or 1.43 – has a drastic effect on the behavior of the green branch.

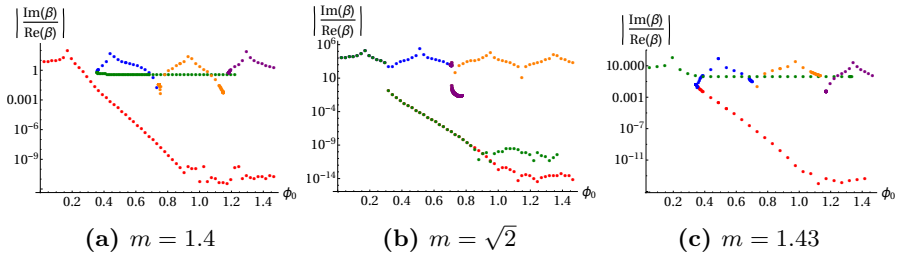


Figure 5.5: The classicality conditions in terms of the asymptotic variables, (5.50). From left to right the ratios of the imaginary part of β to its real part are plotted in function of ϕ_0 for three different masses 1.4, $\sqrt{2}$ and 1.43. The colors are the same as in Figure 5.2.

At a technical level, the discrepancy between the classical predictions of Ψ and $\tilde{\Psi}$ can be traced to the generating function. In particular, the comparison shows that the seemingly classical behavior of the new solutions in terms of the usual variables is entirely due to the growing phase factor in the usual NBWF, which is a universal surface term that is the generating function. This is absent in $\tilde{\Psi}$ – and in the dual partition function – and the new classicality conditions are not sensitive to this. Instead they are more stringent and depend on the interior dynamics and on the quantum state, which are encoded asymptotically e.g. in β . Therefore, they appear more physical from a dual perspective.

However we should emphasise that classicality conditions derived à la WKB are inherently approximate¹⁴. They are only a sufficient condition for classical evolution to be predicted. It is therefore possible that classical evolution holds in the regime where the new, more stringent set of classicality conditions break down. To verify this, however, one would need to evolve the entire wave function. Alternatively, it is also possible that the breakdown of classicality conditions is an indication of large quantum effects, possibly induced by the scalar field in a regime which exhibits features of eternal inflation. This would mean that in fact the inequality in the old classicality conditions must be made stronger. It would be very interesting to explore this question further.

The last figure of this section shows the relative probabilities of the different classical histories predicted by the NBWF, which are given by $\sim e^{-2\text{Re}(I)}$. In general the solution with the most negative real part of the on-shell action

¹⁴ They are an inequality and it is not clear how precisely their formulation in different bases should be related.

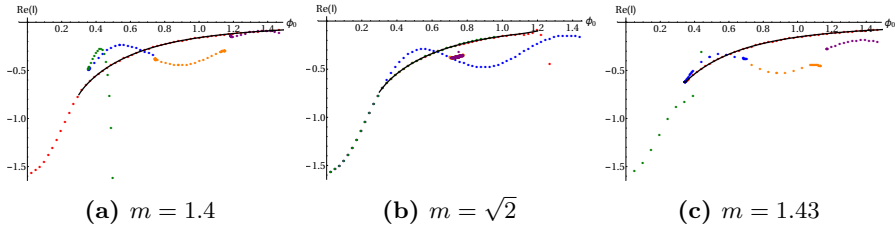


Figure 5.6: The real part of the action versus ϕ_0 , the absolute value of the scalar field at the SP. This specifies the relative probabilities of the solutions shown in Figure 5.2. The colors coincide with the colors used there. For the case $m^2 = 2$ the red and the green branch coincide. The full black curve shows the solutions that obey both sets of classicality conditions.

provides the dominant contribution to the wave function. In Figure 5.6 we show the real part of the action for each family of saddle points, whether classical or not. Notice that for $m^2 = 2$ the green and the red branch coincide perfectly as a consequence of the enhanced symmetry. Figure 5.6 shows that the action of the new solutions is of the same order of magnitude as the previously known solutions in red¹⁵. At first sight it seems non-trivial to decide which saddle point dominates for a given ϕ_0 .

However the new set of asymptotic classicality conditions selects a unique saddle point solution for each value of α . Specifically Figure 5.5 shows that the only classical saddle point solutions are on the red branch for sufficiently large ϕ_0 . These solutions are denoted with a full black line in Figure 5.6. Conditioning on asymptotically classical behavior, therefore, restores the NBWF prediction of a one-parameter set of classical homogeneous isotropic universes with relative probabilities favoring a low amount of scalar field driven inflation.

5.6 Conclusion

We have derived a sufficient set of conditions on the Euclidean boundary theory in dS/CFT for it to predict classical, Lorentzian bulk evolution for large spatial volumes. The conditions amount to the requirement that the vevs corresponding to the external sources in the dual partition function must be approximately real. This in turn leads to a restriction on the values of the sources and on the path integral defining the partition functions if the dual theory is to be

¹⁵ The results shown in Figure 5.2 for the green branch of solutions with $m = 1.4$ are numerically unstable. We do not expect the apparent divergence to be physical, which should thus not be interpreted as a non-normalizable direction of the probability density.

compatible with the asymptotic semiclassical structure implied by the WDW equation.

To derive the new set of classicality conditions, we first expressed the bulk wave function for large spatial volumes in terms of the sources of the dual partition function. This enabled us to put forward a sharper formulation of dS/CFT in which the wave function of the universe is directly related to the dual partition functions.

The conditions under which the boundary theory predicts classical bulk evolution are stronger than the criteria usually employed in quantum cosmology. We illustrated this in a minisuperspace model comprising homogeneous isotropic histories in gravity coupled to a scalar field, where we identified several families of histories which are predicted to behave classically according to the old classicality conditions but do not obey the new conditions. Besides a number of exotic histories in which the scalar field is large, these also include histories which are relatively small perturbations of empty de Sitter in which the scalar field is small everywhere. This appears to be a generalization to light scalars of the prediction [147] in the Hartle-Hawking state for heavy scalars that empty de Sitter is an isolated point in the ensemble of asymptotically classical histories.

We restricted the discussion in this paper to the NBWF for 4-dimensional dS for the sake of clarity and because the connection between dS and AdS representations of saddle points was worked out explicitly in [150]. Generalization to other spacetime dimensions will be the subject of further research. The canonical transformation can be performed for higher dimensions, as has been worked out in [125] for AdS. It would furthermore be interesting to see if the basis change we apply is also valid for other wave function proposals, such as the tunneling wave function [166, 162].

Chapter 6

Higher spin holography

De Sitter holography was conjectured and analyzed in [18, 20, 19] shortly after the breakthrough of AdS/CFT. The basic objects on both sides of the duality were identified, and their symmetries were compared. We have outlined and illustrated these in the previous chapters.

However, it took another decade before an explicit model was found [21]. The CFT in this proposal is the free Euclidean theory of a large number N of anti-commuting scalar fields. The theory is invariant under the symplectic group $Sp(N)$, under which the fields transform in the vector representation. According to the conjecture in [21], the $Sp(N)$ singlet sector of this CFT is dual to a theory of dynamical gravity, as well as a scalar field and an infinite tower of higher-spin fields on asymptotically de Sitter space. These bulk higher-spin theories were developed by Fradkin and Vasiliev [167, 168, 169, 170, 171], who were able to find interacting equations of motion for the full set of fields.

An important inspiration and set of consistency checks for this dS/CFT conjecture was a corresponding example of AdS/CFT, first conjectured by Klebanov and Polyakov [10]. This proposal had not been derived from string theory, but was instead proposed as a “simple model” for holography, namely one in which the CFT is free. In this case, the CFT is Lorentzian and contains commuting scalar fields and an $O(N)$ gauge group. The corresponding bulk theories are AdS higher-spin theories¹ with the same field content as described

¹ The conjecture [10] was not the first holographic model to contain elements of higher spin theory [172, 173, 174, 175, 176], but it was the first to implement the ideas of [177] to shift attention from matrix-like degrees of freedom to theories with only vector like ones, thereby isolating the particular higher-spin sector described by Vasiliev theory.

above. In fact, these higher spin models were first developed in AdS, before the breakthrough of holography.

In this chapter we will first introduce the CFT of the Klebanov-Polyakov model [10]. We will outline the operator content of the CFT and link it to the corresponding higher spin fields in AdS. In §6.2 we will review the Anninos-Hartman-Strominger (AHS) model of dS/CFT [21]. This chapter will conclude with a review of some of the calculations of the dS wave function that have been performed using this model. We will specifically focus on an observation of [27], which suggests a problem with the normalizability of the wave function.

6.1 Higher spin Anti-de Sitter holography

Each of the examples of holography derived from string theory mentioned in Chapter 2 contain Yang-Mills theory on the CFT side of the duality. Open strings and stacks of branes naturally give rise to $SU(N)$ gauge sectors.

A fundamentally different type of duality was conjectured by Klebanov and Polyakov in [10]. The CFT in this proposal is not a theory of matrices – as are the aforementioned $N \times N$ gauge fields in the adjoint representation of $SU(N)$ – but instead of N massless scalar fields ϕ^A that transform in the fundamental (vector) representation of a gauge group. There are two versions of the model, one with real scalar fields which transform as a vector under an $O(N)$ gauge symmetry, and a model with complex scalars, which has $U(N)$ as the gauge symmetry. In both models, the symmetry group is gauged so only the singlets are physical fields. We will write equations for the $U(N)$ model, but the $O(N)$ case is trivially obtained by identifying $\bar{\phi} \rightarrow \phi$.

The free part of the action contains only the canonical kinetic terms (no mass terms are allowed since they would break scale-invariance)

$$S_0 = \frac{1}{2} \int d^d x \, \eta^{\mu\nu} \delta_{AB} \partial_\mu \bar{\phi}^A(x) \partial_\nu \phi^B(x) , \quad (6.1)$$

which yields the equations of motion $\nabla^2 \phi^A = 0$. This action is invariant under conformal transformations for which the scalars transform as primaries with weight $\Delta_\phi = d/2 - 1$. To describe the gauge-invariant sector of the theory, consider $U(N)$ (or $O(N)$) singlets formed from bilinear combinations of the elementary fields. For example, the combination

$$J^{(0)}(x) \equiv : \bar{\phi}_A(x) \phi^A(x) : = \lim_{y \rightarrow x} [\bar{\phi}_A(x) \phi^A(y) - \langle \bar{\phi}_A(x) \phi^A(y) \rangle] , \quad (6.2)$$

with the index lowered using δ_{AB} , is a scalar both under rotations as well as gauge transformations. Furthermore, it transforms under dilatations as

$(\hat{D}\bar{\phi}_A)\phi^A + \bar{\phi}_A(\hat{D}\phi^A) = (x \cdot \partial_x + 2\Delta_\phi)J^{(0)}$ and at the origin it is annihilated by the special conformal transformations, $\hat{K}_\mu J^{(0)}(0) = 0$. That is to say, it is a primary scalar operator of the theory.

Similarly, the $U(N)$ model contains the following real bilinear with one derivative (in the $O(N)$ model, this combination vanishes)

$$J_\mu^{(1)}(x) \equiv i : (\bar{\phi}_A(x)\partial_\mu\phi^A(x) - \partial_\mu\bar{\phi}_A(x)\phi^A(x)) : . \quad (6.3)$$

This current transforms under dilatations with weight $2\Delta_\phi + 1 = d - 1$ and in the origin it is annihilated by special conformal transformations,

$$\hat{K}_\nu J_\mu^{(1)}(0) = i \left[\bar{\phi}_A(x)\partial_\mu\hat{K}_\nu\phi^A(x) - \partial_\mu\hat{K}_\nu\bar{\phi}_A(x)\phi^A(x) \right]_{x=0} = 0 , \quad (6.4)$$

where we have used (2.31) to show that $\partial_\mu\hat{K}_\nu\phi\Big|_{x=0} = 2\delta_{\mu\nu}\Delta_\phi$. Therefore, it transforms as a primary operator under conformal transformations.

There are primary conserved currents $J_{\mu_1\dots\mu_s}^{(s)}$ of spin s for every positive integer s in the $U(N)$ model and for every even positive integer s in the $O(N)$ model. As a first step, consider the following conserved (but not necessarily primary) currents (with normal ordering understood)

$$\tilde{J}_{\mu_1\dots\mu_s}^{(s)} \equiv \bar{\phi}_A \overset{\leftrightarrow}{\partial}_{\mu_1} \dots \overset{\leftrightarrow}{\partial}_{\mu_s} \phi^A , \quad \overset{\leftrightarrow}{\partial}_\mu \equiv \frac{\overset{\rightarrow}{\partial}_\mu - \overset{\leftarrow}{\partial}_\mu}{2} . \quad (6.5)$$

These are $U(N)$ singlet currents which are conserved by the equations of motion, $\partial^{\mu_1}\tilde{J}_{\mu_1\dots\mu_s}^{(s)} = 0$. They are symmetric in their spacetime indices. By Noether's theorem, there is a corresponding symmetry for each s . Next, consider improvement terms of the form

$$\tilde{D}_{(\mu_1\mu_2\dots\mu_{n-1}\mu_n)}\tilde{J}_{\mu_{n+1}\dots\mu_s}^{(s-n)} , \quad \tilde{D}_{\mu\nu} \equiv \frac{1}{4}(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu) , \quad (6.6)$$

for even $n \leq s$. Each of these terms is conserved because \tilde{J} is conserved and $\partial^\mu\tilde{D}_{\mu\nu} = 0$. Adding such terms to the currents $\tilde{J}^{(s)}$, we can change their trace while preserving their conservation. For the case $s = 2$, we can use this method to find the improved stress-energy tensor of the theory, which is traceless and conserved

$$\begin{aligned} J_{\mu\nu}^{(2)} &= \tilde{J}_{\mu\nu}^{(2)} + \frac{1}{d-1}\tilde{D}_{\mu\nu}(\bar{\phi}_A\phi^A) \\ &\propto \partial_{(\mu}\bar{\phi}_A\partial_{\nu)}\phi^A - \frac{1}{4(d-1)}[(d-2)\partial_\mu\partial_\nu + \eta_{\mu\nu}\partial^2](\bar{\phi}_A\phi^A) , \end{aligned} \quad (6.7)$$

We can build the conserved charges corresponding to conformal symmetries from this stress tensor as in §2.3.2. From the general argument in §2.3.4, it then follows that all operators in the theory can be written as linear combinations of primaries and their descendants. The currents (6.5) cannot be written as descendants of primaries of lower s , so there must be a primary current for each value of s .

To find their explicit form, we can use the method of [178, 179] and define a polynomial for each s ,

$$\begin{aligned} J^{(s)}(x, z) &\equiv J_{\mu_1 \dots \mu_s}^{(s)} z^{\mu_1} \dots z^{\mu_s} \\ &\equiv D^{(s)}(z \cdot \partial_v, z \cdot \partial_w) \bar{\phi}_A(v) \phi^A(w) \Big|_{v=x=w}, \end{aligned} \quad (6.8)$$

where $D^{(s)}(y, \bar{y})$ is a polynomial of degree s . Since $J_{\mu_1 \dots \mu_s}^{(s)}$ is traceless, it is sufficient to restrict to the polynomials of a (complex) null vector z , i.e. with $\eta_{\mu\nu} z^\mu z^\nu = 0$. For this operator to be primary, we want to choose $D^{(s)}$ in such a way that the special conformal transformations annihilate $J^{(s)}$ in the origin. We can use the identities

$$\begin{aligned} D^{(s)}(y, \bar{y}) \Big|_{(y, \bar{y})} z^i \partial_v^i \bar{\phi}_A(v) \phi^A(w) &= y D^{(s)}(y, \bar{y}) \Big|_{(y, \bar{y})} \bar{\phi}_A(v) \phi^A(w), \\ D^{(s)}(y, \bar{y}) \Big|_{(y, \bar{y})} v_i \bar{\phi}_A(v) \phi^A(w) &= (v^i + z^i \partial_y) D^{(s)}(y, \bar{y}) \Big|_{(y, \bar{y})} \bar{\phi}_A(v) \phi^A(w), \end{aligned} \quad (6.9)$$

where $|_{(y, \bar{y})}$ is an abbreviation for $|_{(y=z \cdot \partial_v, \bar{y}=z \cdot \partial_w)}$, to translate the requirement $K_i J^{(s)}(0) = 0$ into a differential equation for $D^{(s)}$,

$$[\Delta_\phi(\partial_y + \partial_{\bar{y}}) + y \partial_y^2 + \bar{y} \partial_{\bar{y}}^2] D^{(s)}(y, \bar{y}) = 0. \quad (6.10)$$

For this equation can be solved in terms of the Gegenbauer polynomials

$$D^{(s)}(y, \bar{y}) = (y + \bar{y})^s C_s^{(\Delta_\phi - \frac{1}{2})} \left(\frac{y - \bar{y}}{y + \bar{y}} \right). \quad (6.11)$$

For $\Delta_\phi = \frac{d}{2} - 1$, and $d = 3$ all $s > 1$ Gegenbauer polynomials vanish. Non-trivial primary currents can nevertheless be generated with the regularized Gegenbauer polynomials $C_n(x) = \lim_{m \rightarrow 0} C_n^{(m)}(x)/m$.

To conclude this part, we can obtain a generating function for the primary currents of all s by using the generating function for Gegenbauer polynomials

$(1 - 2xt + t^2)^{-\alpha} = \sum_{s \geq 0} C_s^{(\alpha)}(x) t^s$, and summing up all polynomials in (6.8)

$$\begin{aligned} J(x, z) &\equiv \sum_{s \geq 0} J^{(s)}(x, z) \\ &= [1 - 2(y - \bar{y}) + (y + \bar{y})^2]^{\frac{1}{2} - \Delta_\phi} \Big|_{(y, \bar{y})} \bar{\phi}_A(v) \phi^A(w) \Big|_{v=x=w} . \end{aligned} \quad (6.12)$$

Again, the case $d = 3$ is degenerate. We can divide the generating function by $\Delta_\phi - \frac{1}{2}$ and take the limit to zero. The expression for the scalar primary current diverges in that limit. It is possible to renormalize it by subtracting it before taking the limit, and adding it back by hand afterwards. The result of this procedure is

$$J(x, z) = [1 + \log(1 - 2(y - \bar{y}) + (y + \bar{y})^2)]_{(y, \bar{y})} \bar{\phi}_A(v) \phi^A(w) \Big|_{v=x=w} . \quad (6.13)$$

If this CFT is to have a holographic dual, each of these single-trace operators would correspond to a spin- s gauge field propagating on AdS_{d+1} : a tensor with s spacetime indices $\Phi_{\mu_1 \dots \mu_s}^{(s)}$, symmetric in their indices, that transforms to first order around empty AdS as a derivative under a linearized gauge symmetry generated by a tensor with $s - 1$ indices.

$$\delta \Phi_{\mu_1 \dots \mu_s}^{(s)} = \nabla_{(\mu_1} \xi_{\mu_2 \dots \mu_s}^{(s-1)} . \quad (6.14)$$

The fields are “double-traceless” [180]: $\nabla^{\mu_1} \nabla^{\mu_2} \Phi_{\mu_1 \dots \mu_s}^{(s)} = 0$, and the parameters $\xi_{\mu_1 \dots \mu_{s-1}}^{(s-1)}$ are simply traceless. Theories with these fields exist for both even and odd values of s . For example there is a scalar $\Phi^{(0)}$ which is not associated with a gauge symmetry, a vector $\Phi_\mu^{(1)}$ which transforms like the vector potential in electromagnetism $\delta \Phi_\mu^{(1)} = \partial_\mu \xi^{(0)}$, and a graviton $\Phi_{\mu\nu}^{(2)}$ for which the transformations (6.14) are diffeomorphisms $\delta \Phi_{\mu\nu}^{(2)} = \partial_\mu \xi_\nu^{(1)} + \partial_\nu \xi_\mu^{(1)}$. There is a space gauge-invariant action – due to Fronsdal [181] – that gives rise to free gauge-invariant equations of motion for every s . They take the form [175, 182]

$$\begin{aligned} & -(\nabla_\rho \nabla^\rho - m^2) \Phi_{\mu_1 \dots \mu_s}^{(s)} + s \nabla_{(\mu_1} \nabla^\rho \Phi_{\mu_2 \dots \mu_s \rho)}^{(s)} \\ & - \frac{s(s-1)}{2(d+2s-3)} g_{(\mu_1 \mu_2} \nabla^\nu \nabla^\rho \Phi_{\mu_3 \dots \mu_s \nu \rho)}^{(s)} = 0 , \end{aligned} \quad (6.15)$$

with $m^2 = (s-2)(d+s-3) - 2$. In the gauge $\nabla^\nu \Phi_{\nu \mu_2 \dots \mu_s}^{(s)} = 0$, this equation of motion reduces to [182]

$$(\nabla_\rho \nabla^\rho - m^2) \Phi_{\mu_1 \dots \mu_s}^{(s)} = 0 . \quad (6.16)$$

After working out the covariant derivatives in this equation, this equation can be seen to have solutions that go like $z^{d-\Delta-s}$ near $z \approx 0$, with $\Delta = d - 2 + s$.

The putative holographic dual to the simple CFT (6.1) must contain an infinite tower of higher-spin fields, one for each nonnegative s in case of the $U(N)$ model and one for each even s for $O(N)$. The fact that the CFTs are free would not rid the bulk theory of interactions. Consider for example the three-point function of the scalar primary current. By subtracting the disconnected part, we get the normal-ordered three point function

$$\langle : \bar{\phi}_A(x) \phi^A(x) :: \bar{\phi}_B(y) \phi^B(y) :: \bar{\phi}_C(z) \phi^C(z) : \rangle = \frac{N}{|x-y||y-z||z-x|} . \quad (6.17)$$

In the bulk, such a correlation function appears only if the scalar interacts non-trivially with itself. In general, the non-vanishing connected CFT correlation functions correspond to bulk interactions.

A valid question is whether such an interacting theory of higher spin fields can be self-consistent. Especially in flat space several arguments can be made against higher-spin particles that interact with “low-spin matter”.² Consistent interacting theories containing an infinite tower of higher-spin fields do exist on (Anti) de Sitter space. In particular, Fradkin and Vasiliev established a self-consistent and fully interacting set of classical equations of motion [167, 168, 169, 170]. (See [188, 189] for more recent reviews.) As in the CFT there is a minimal model which contains one massless spin- s field for each nonnegative even s , and a non-minimal model which has such a field in its spectrum for each nonnegative integer s .

It was observed in [10] that the four-dimensional minimal Vasiliev model has the correct field content to be dual to the three-dimensional $O(N)$ model

² More specifically, there are tight constraints on higher-spin interactions in flat space coming from the following no-go theorems.

- The Weinberg low-energy theorem [183] implies that a theory with an S-matrix cannot have couplings to massless higher-spin fields in the low-energy limit.
- The Coleman-Mandula theorem and its supersymmetric generalization [46, 47] require asymptotic higher-spin charges to vanish. That is, even if a massless higher-spin field exists, that field will come with an associated higher-spin gauge symmetry which must become trivial at long distances. Indeed such a charge would carry spacetime indices, i.e. it would extend the Poincaré algebra in violation of the Coleman-Mandula theorem.
- The Weinberg-Witten theorem [184] states that theories containing a spin $s > 1$ particle cannot have a local energy-momentum tensor. More specifically it was shown in [185, 186] that spin $s > 2$ particles cannot couple minimally to gravity in an asymptotically flat spacetime.

A more careful explanation of each of these no-go theorems and their limitations is presented for example in [187].

and the $4d$ nonminimal Vasiliev model has the right spectrum to be dual to the $U(N)$ vector CFT in $3d$. The fall-offs fields are consistent with the weights $\Delta = s + 1$ of the higher-spin primary currents for $d = 3$. The bulk scalar is the conformally coupled one, with $m^2 = -2$ to be dual to an operator of weight $\Delta_{\phi^A \phi^A} = 1$, albeit in “alternate quantization” [190] where the CFT operator (and not the source) is dual to the leading fall-off. The authors of [10] conjectured that the CFT described by (6.1) is holographically dual to the nonminimal Vasiliev model in AdS_4 and they proposed that its real variant could be dual to the minimal model.

They furthermore considered the CFT obtained by applying the relevant deformation $\lambda(\bar{\phi}_A \phi^A)^2/2N$ to free action (6.1) and following the resulting renormalization group (RG) flow to the IR fixed point, which is of the Wilson-Fisher type. This interacting $U(N)$ (or $O(N)$) model can be obtained through a Legendre transformation: by introducing an auxiliary field σ to rewrite the action

$$\begin{aligned} S_{\text{crit}} &= \frac{1}{2} \int d^d x \left[\partial_\mu \bar{\phi}_A \partial^\mu \phi^A + \frac{\lambda}{N} (\bar{\phi}_A \phi^A)^2 \right] \\ &= \frac{1}{2} \int d^d x \left[\partial_\mu \bar{\phi}_A \partial^\mu \phi^A + 2\sigma \bar{\phi}_A \phi^A - \frac{N}{\lambda} \sigma^2 \right]_{\sigma=\sigma_*}, \end{aligned} \quad (6.18)$$

where $\sigma_* = \lambda(\bar{\phi}_A \phi^A)/N$ is the on-shell value of σ . In the path integral expression of the partition function, this is achieved by integrating over σ as well. It is then possible to integrate out the $\bar{\phi}$ and ϕ fields altogether and derive an effective action for σ [191, 10]. The scaling dimension of σ is $d - 1$, which for $d = 3$ is the other conformal weight that can be dual to an AdS_4 scalar of $m^2 = -2/l^2$. Therefore, Klebanov and Polyakov conjectured that the critically interacting $3d$ vector models are dual to *the same* Vasiliev theory in AdS , but with the standard boundary conditions: the CFT operator σ is dual to the subleading ($\sim z^2$) fall-off of the bulk scalar.

These conjectures were checked and generalized in subsequent work. In particular, all three-point functions were worked out in [182]. Furthermore, the original model of [10] was found to be part of a one-parameter family of vector models coupled to Chern-Simons theory, dual to a one-parameter family of higher-spin theories [192].

6.2 The Anninos-Hartman-Strominger model

Many models of AdS/CFT obtained from string theory contain CFT gauge operators in the adjoint representation of $U(N)$. These models contain operators

of infinitely high dimension with only a single trace loop over the gauge indices. For example, one can consider scalar operators of the form $\mathcal{O}_A^B \mathcal{O}_B^C \dots \mathcal{O}_D^A$. An obstruction to using such models in dS/CFT is that they are dual to increasingly tachyonic fields, given the relation (4.56) between dimensions of scalar CFT operators and the mass of bulk fields. This problem does not occur for the vector model described in the previous section, where all single trace primary operators correspond to gauge fields in Vasiliev theory. Furthermore, Vasiliev theory does not only exist on AdS, but has also been worked out on de Sitter space [171]. One may therefore expect to find analogs of the Klebanov-Polyakov model in dS.

In [21] it was proposed that a CFT with gauge group $Sp(N)$ (for even N) and *anticommuting* scalars is dual to Vasiliev in dS₄. The correlation functions of the CFT are given by the path integral weighted by the exponent of the Euclidean action

$$I = \int d^3x \, \delta^{ij} \Omega_{AB} \partial_i \chi^A \partial_j \chi^B, \quad \Omega_{AB} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad (6.19)$$

where $\mathbb{1}$ is the $\frac{N}{2} \times \frac{N}{2}$ identity matrix. The equations of motion are again $\nabla^2 \chi^A = 0$ and its complex conjugate. The construction of traceless conserved currents is analogous to the $O(N)$ model in AdS, with $\bar{\chi}_A = \Omega_{AB} \bar{\chi}^B$. In particular, there is a traceless conserved stress-energy tensor. The conformal weight of χ is again $1/2$. A similar continuation is possible for the $U(N)$ model [193], which maps onto the same action (6.1) but with ϕ replaced by an anticommuting complex scalar χ . In particular, the gauge group of the fermionic model remains $U(N)$.

The correlation functions of this free $Sp(N)$ theory can be obtained from the $O(N)$ correlation functions upon analytic continuation $N \rightarrow -N$. This can be seen at the level of the CFT partition function expressed as a path integral. In this case, the action is deformed by sources coupling to the higher-spin currents $J^{(s)}$ of (6.8). We will denote this schematically as $\Delta I = \sigma_{(s)} \cdot J^{(s)}$. The partition function is then a Gaussian path integral, both in the $O(N)$ and in the $Sp(N)$ model, which can be calculated exactly

$$Z[\sigma_{(s)}] = \det(-\nabla^2 + \sigma_{(s)} \tilde{D}^{(s)})^{\mp N}. \quad (6.20)$$

The exponent is $-N$ for the $O(N)$ model (it is the result of a Bosonic Gaussian integral) and $+N$ for the $Sp(N)$ model (due to the Fermi statistics of the CFT fields χ^A , it is a Gaussian Berezin integral [194]).

The free $Sp(N)$ vector CFT can be deformed by a relevant operator – in this case $\lambda(\Omega_{AB} \chi^A \chi^B)^2/2N$ – to trigger RG flow towards the critically interacting IR fixed point. It was shown in [21] that the correlation functions in this theory can again be obtained from the critical $O(N)$ model by continuing $N \rightarrow -N$.

On the bulk side, Vasiliev theory in dS is related by analytic continuation to AdS [195]. Since the number of operators in the vector $O(N)$ and $U(N)$ CFTs scale linearly with N , the holographic dictionary for the cosmological constant of Vasiliev theory is $G_N \Lambda \sim N^{-1}$. Therefore, the AdS-Vasiliev/ $O(N)$ conjecture and the nontrivial consistency checks done in [182] can be extended to a dS-Vasiliev/ $Sp(N)$ conjecture, thereby providing an explicit implementation of the dS/CFT proposal conjectured almost a decade before.

Several other models of dS/CFT were conjectured in [193]. These are related to the one-parameter family of AdS/CFT vector models with Chern-Simons couplings mentioned at the end of the previous section.

6.3 The holographic wave function of Vasiliev's universe

The partition function (6.20) provides a direct³ handle on the implications of this holographic proposal. There are an infinite number of deformations, parameterized by three-dimensional fields $\sigma_{(s)}$, for each (even) nonnegative integer s . Some of the directions were first explored in [27], in particular the direction in which only $\sigma_{(0)}$ is nonzero and constant on the manifold S^3 of the CFT. Their result is shown in Figure 6.1. While empty de Sitter space is a local maximum of the partition function – as one might expect if it is to be interpreted as a wave function that describes a de Sitter universe that is at least perturbatively stable – it is not the unique maximum. There are an infinite number of other maxima at negative values of $\sigma_{(0)}$, at which the partition function is exponentially large: $\log(Z) \propto -\sigma_{(0)}$.

This result suggests that the direction of constant $\sigma_{(0)}$ is a non-normalizable direction of the wave function. To confirm or refute the hypothesis of non-normalizability, it is necessary to get a better idea of the integration manifold. The direction of spatially constant $\sigma_{(0)}$ is part of an infinite-dimensional space of deformations. These boundlessly growing oscillations may still integrate to a finite number if they are sufficiently “thin” in other directions.

Furthermore, it is not clear which exact integration measure has to be used. For a quantum theory with a Hilbert space, this is fixed by the inner product and the complete set of states which is used to define the wave function. When a classical theory is canonically quantized, this integration measure on the Lagrangian submanifold on which the wave function takes values can be derived

³ The calculation is not straightforward, because the result generically diverges. It is necessary to renormalize the determinant to get a sensible answer.

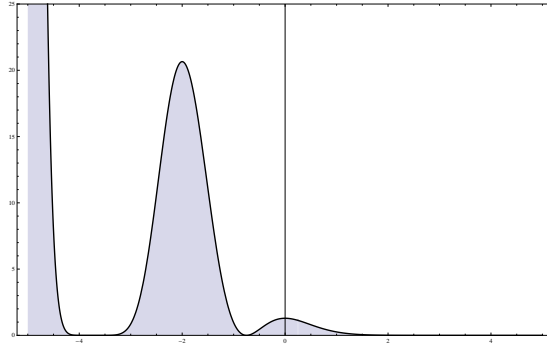


Figure 6.1: Result obtained in [27] for the regularized partition function of the CFT (6.19) for $N = 2$. The vertical axis is the logarithm of $|Z_{\text{finite}}|^2$, as a function of the constant mass deformation $\sigma_{(0)}$ on the horizontal axis.

from the phase space symplectic form. However, dS/CFT does not clarify what the bulk Hilbert space is (it does not correspond to a CFT Hilbert space, since the Euclidean CFT defined by (6.19) does not satisfy the reflection-positivity condition to have a positive inner product and a Hilbert space). Furthermore, the dS/CFT relation (4.57) only defines the quantity ψ without clarifying what the corresponding classical phase space is. It is therefore not clear what probability density this “wave function” actually predicts.

Chapter 7

Bulk-local operators

If quantum gravity in de Sitter space is defined by a Euclidean CFT, and if there is a regime where field theory on a fixed dS background is a good approximation, then there should be operators in the CFT which (in that approximation) correspond to the local bulk field operators of canonical QFT. It should be possible to calculate all bulk correlation functions of local operators as CFT correlation functions. For a scalar field, this should connect the dS/CFT proposal to the canonical quantization of a scalar field on a fixed planar patch background, cf. §4.2.2, at least in the regime where the backreaction of the fields is negligible.

In this chapter, we will use the correspondence (4.55) between conformal symmetries in a Euclidean CFT and isometries in de Sitter space to find the most general CFT operator that transforms as a local bulk scalar under this symmetry group. The method used in this analysis is similar to [196], which contributes to a bulk-reconstruction program in AdS/CFT initiated in [197, 198, 199, 200].

As anticipated in §4.5, we expect to gain insight into the emergence of the bulk time direction, and bulk locality in general, in dS/CFT. We will, however, find that the CFT does not exactly reproduce the results of canonical quantization. The operators we find have similar bulk time dependence as the scalar field modes in (4.22), but the correlation functions we find are crucially different. We trace this difference back to the fact that the correlation functions of the CFT, defined as a Euclidean path integral, cannot account for the canonical commutator $[\hat{\phi}, \hat{\pi}]$, where $\hat{\phi}$ is a bulk field operator and $\hat{\pi}$ is its conjugate momentum.

7.1 Generic bulk scalar

We start our analysis in de Sitter space of arbitrary dimensions dS_{d+1} . Bulk scalar fields are characterized by their transformations under the dS isometry group (4.2). The same group is represented on primary operators in the Euclidean CFT. According to the dS/CFT proposal, these representations are related by (4.55). We will use the fact that every operator in a CFT can be written as a linear combination of primary operators and their descendants, to find the map between primary operators in the CFT and local scalar operators on de Sitter space.

7.1.1 Planar patch

In the planar patch, the above can be expressed as a sum of terms of the form

$$\Phi(\eta, \vec{x}) = \sum_{\alpha} \int d^d y \, G_{\alpha}(\eta, \vec{x}; \vec{y}) \mathcal{O}_{\alpha}(\vec{y}) , \quad (7.1)$$

where α ranges over all conformal primaries with a weight that satisfies $\Delta(d - \Delta) = m^2$, where G_{α} is a smearing function that will be determined and where \vec{y} ranges over the manifold on which the CFT is defined, i.e. \mathbb{R}^d . To be a bulk scalar field, $\Phi(\eta, \vec{x})$ should be invariant under the rotations and boosts that leave the spacetime point (η, \vec{x}) invariant. For example, consider the scalar field operator in the point $\eta = -1$, $\vec{x} = 0$, cf. Figure 7.1. It must be invariant under the spatial rotations around the origin of the static patch. Through the equivalence (4.55), this is represented by the rotation operator \hat{M}_{ij} in (2.27) that acts as a differential operator on the primaries in the CFT,

$$\begin{aligned} 0 = \hat{L}_{ij} \Phi(-1, 0) &= \int d^d y \, G(-1, 0; \vec{y}) \hat{M}_{ij} \mathcal{O}(\vec{y}) \\ &= \int d^d y \, G(-1, 0; \vec{y}) (y_i \partial_j - y_j \partial_i) \mathcal{O}(\vec{y}) . \end{aligned} \quad (7.2)$$

Integrating by parts, this requirement implies that G can only depend on $y \equiv |\vec{y}|$. Similarly, boosts L_{0i} have to leave the local scalar field operator invariant,

$$\begin{aligned} 0 = \hat{L}_{0i} \Phi(-1, 0) &= \int d^d y \, G(-1, 0; y) \frac{\hat{P}_i - \hat{K}_i}{2} \mathcal{O}(\vec{y}) \\ &= \int d^d y \, \mathcal{O}(\vec{y}) \frac{y_i}{y} [2(d - \Delta)y - (1 - y^2)\partial_y] G(-1, 0; y) . \end{aligned} \quad (7.3)$$

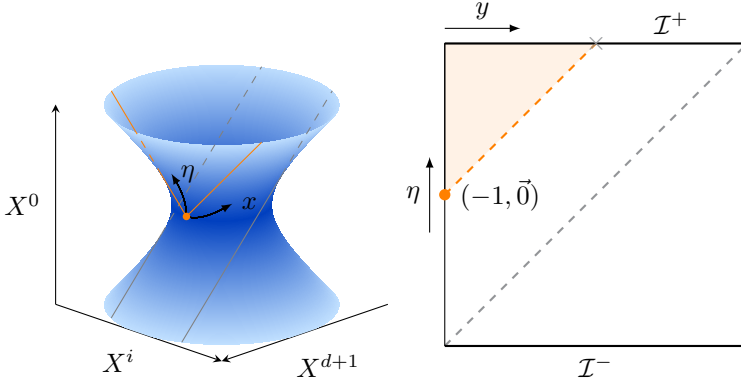


Figure 7.1: Two illustrations of the bulk-local operator corresponding to (7.4). On the left is de Sitter space as a hyperboloid in $d + 2$ -dimensional Minkowski space. The planar patch coordinates are given at their “origin” $\eta = -1$, $\vec{x} = 0$. This point corresponds to the point $X^0 = 0$, $X^i = 0$ and $X^{d+1} = -l_{\text{dS}}$ in the embedding space. The future light-cone emanating from the origin is indicated in orange, and the past causal horizon is depicted by the diagonal gray lines. The figure on the right is the corresponding Penrose diagram with the same information. The causal future of the planar patch origin corresponds to the shaded orange region. The infinite future \mathcal{I}^+ and past \mathcal{I}^- are at finite distance in this conformal diagram.

This constrains G to be proportional to $(1 - y^2)^{\Delta-d}$.

As it stands, G is generically singular at $y = 1$ because bulk scalars of any positive mass correspond to CFT operators with $\text{Re}(\Delta) < d$. Therefore, the previous result is not well-defined. The problematic points are those in the infinite future of dS which are light-like separated from $\eta = -1$, $\vec{x} = 0$. In other words, it is the intersection of its future light-cone with the conformal boundary \mathcal{I}^+ . This locus is indicated with “ \times ” on the Penrose diagram in Figure 7.1. Outside of the light-cone emanating from the bulk point, i.e. for $y > 1$, G has a branch cut (except when Δ is an integer. We will revisit that situation in the next section). The singularity can be resolved by an “ $i\delta$ -prescription”,¹ for example by adding a contribution $\pm i\delta$ with $\delta > 0$ and taking the limit $\delta \rightarrow 0$,

$$G_{\pm}(-1, 0; y) = \gamma(1 - y^2 \pm i\delta)^{\Delta-d} = \gamma e^{\pm i\pi(\Delta-d)}(y^2 - 1 \mp i\delta)^{\Delta-d}, \quad (7.4)$$

where γ is a constant of proportionality that will be determined later. In the rightmost equation, we have rotated the branch cut by adding a phase $e^{-i\theta}$ in between the brackets, a phase $e^{i\theta(\Delta-d)}$ outside, and sending $\theta : 0 \rightarrow \pm\pi$.

¹ We will not use the conventional letter ϵ to distinguish this prescription explicitly from the $i\epsilon$ prescription in the bulk of de Sitter, which was used in (4.31).

Another way to implement the $i\delta$ -prescription is to deform the y -integral (the radial part of $d^d y$) from the positive real axis to a contour with the same endpoints but a small imaginary part, passing the singularity at $y = 1$ either from above (for $-i\delta$) or below (for $+i\delta$). We can obtain an operator which has support either only inside the intersection of the light-cone with \mathcal{I}^+ or completely outside by taking linear combinations,

$$G_{\text{out}} \equiv G_+ - G_- , \quad G_{\text{in}} \equiv e^{-i\pi(\Delta-d)} G_+ - e^{i\pi(\Delta-d)} G_- . \quad (7.5)$$

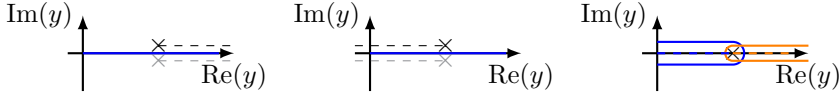


Figure 7.2: The functional dependence of G on the coordinate $y = |\vec{y}|$ can be extended into the complex plane. The singularity at $y = 1$ is indicated by \times , whereas the corresponding branch cut is denoted as a dashed line. The figure on the left-hand side depicts the integration contour along the positive real axis in blue. The singularity and branch cuts are moved off the real axis by the contribution $\pm i\delta$. Analogously, the contour can be slightly deformed off the real axis and the branch cut can stay in place. This is depicted on the right-hand side. The contours can then be deformed without crossing singularities or branch cuts to give the functions G_{out} and G_{in} in (7.5).

To get the result for a general point in the static patch, we can again use the isometries. First, an η -translation from -1 to an arbitrary value η can be achieved in the CFT by the dilation $e^{\log(-\eta)\hat{D}}$, which maps $\mathcal{O}(\vec{y}) \rightarrow (-\eta)^\Delta \mathcal{O}(-\eta \vec{y})$. Afterwards, a translation to \vec{x} gives

$$\begin{aligned} \Phi_\pm(\eta, \vec{x}) &= \sum_\alpha \gamma_\alpha^\pm \int d^d y (1 - y^2 \pm i\delta)^{\Delta-d} (-\eta)^\Delta \mathcal{O}_\alpha(\vec{x} - \eta \vec{y}) \\ &= \sum_\alpha \gamma_\alpha^\pm \int d^d y \left(\frac{\eta^2 - |\vec{y} - \vec{x}|^2}{-\eta} \pm i\delta \right)^{\Delta-d} \mathcal{O}_\alpha(\vec{y}) . \end{aligned} \quad (7.6)$$

We have not been careful to keep track of the normalization of δ since only its sign matters. The integrand in (7.6) can be recognized as the de Sitter invariant length (4.12) where one of the points has been taken to the asymptotic future \mathcal{I}^+ , cf. Figure 7.2. We can therefore rewrite (7.6) in a way that leans itself better to generalizations of coordinate system,

$$\Phi_\pm(\eta, \vec{x}) = \sum_\alpha \gamma_\alpha^\pm \lim_{\eta_0 \rightarrow 0} (-2\eta_0)^{\Delta-d} \int d^d y [P(\eta, \vec{x}; \eta_0, \vec{y}) \pm i\delta]^{\Delta-d} \mathcal{O}_\alpha(\vec{y}) . \quad (7.7)$$

One can check that this result transforms under the conformal algebra as expected, for example under dilatations as $\hat{D}\Phi = (\eta\partial_\eta + x\partial_x)\Phi$.

7.1.2 Fourier space

As we have seen in §4.2.1, it is useful to Fourier transform the local field operators in the spatial directions of the planar patch. This maps local operators $\Phi(\eta, \vec{x})$ onto mode function operators

$$\Phi_{(\pm, \vec{k})}(\eta) = \int \frac{d^d x}{(2\pi)^{\frac{d}{2}}} e^{i\vec{k} \cdot \vec{x}} \Phi_{\pm}(\eta, \vec{x}) , \quad (7.8)$$

which diagonalize the spatial part of the kinetic terms in the scalar field action. Since the functions G_{\pm} only depend on \vec{x} and \vec{y} through their difference, (7.6) is a convolution. Its Fourier transform is just a product of Fourier transforms of G and the CFT operator,

$$\begin{aligned} \Phi_{(\pm, \vec{k})}(\eta) &= \sum_{\alpha} \gamma_{\alpha}^{\pm} \int d^d \vec{z} e^{i\vec{k} \cdot \vec{z}} G_{\pm}(\eta, \vec{z}) \int \frac{d^d y}{(2\pi)^{\frac{d}{2}}} e^{i\vec{k} \cdot \vec{y}} \mathcal{O}_{\alpha}(\vec{y}) \\ &\equiv \sum_{\alpha} \gamma_{\alpha}^{\pm} G_{(\pm, \vec{k})}(\eta) \mathcal{O}_{\vec{k}}^{\alpha} , \end{aligned} \quad (7.9)$$

where $\mathcal{O}_{\vec{k}}^{\alpha}$ is the Fourier transform of the local primaries $\mathcal{O}_{\alpha}(x)$.

To calculate the Fourier transform of G , it is convenient to rewrite (7.6) using a particular form of the Schwinger parameterization. In its original form, it states that for $\beta > 0$ and $\text{Re}(A) > 0$, the $-\beta$ th power of A can be expressed as an integral

$$A^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} \frac{du}{u} u^{\beta} e^{-uA} . \quad (7.10)$$

We can use this with $\beta = d - \Delta$ to rewrite the integrand of (7.6). Since A must have a positive real part, a good identification is $A = \delta \mp i(\eta^2 - z^2)$, where we have again rescaled δ by a positive number. Independent of the sign of $\eta^2 - z^2$, but with $\delta > 0$ and $s \equiv \pm iu$ we get

$$\begin{aligned} G_{(\pm, \vec{k})}(\eta) &= \frac{\gamma(-\eta)^{d-\Delta}}{\Gamma(d-\Delta)} \int_0^{\pm i\infty} \frac{ds}{s} s^{d-\Delta} \int d^d x e^{i\vec{k} \cdot \vec{x}} e^{s(\eta^2 - x^2 \pm i\delta)} \\ &= \frac{\pi^{\frac{d}{2}} \gamma(-\eta)^{d-\Delta}}{\Gamma(d-\Delta)} \int_0^{\pm i\infty} \frac{ds}{s} s^{-\nu} e^{s(\eta^2 \pm i\delta) - \frac{k^2}{4s}} , \end{aligned} \quad (7.11)$$

where $\nu = \Delta - \frac{d}{2}$. This result has the form of the integral representation of the Hankel functions, where for ease of notation we use $H^+ = H^{(1)}$ and $H^- = H^{(2)}$,

$$H_\nu^\pm(x) = \pm \frac{1}{i\pi} \int_{-\infty}^{\infty \pm i\pi} dt e^{-\nu t + x \sinh t}, \quad e^t = \frac{2}{k}(-\eta \pm i\delta)s. \quad (7.12)$$

Therefore, these modes are proportional to Hankel functions, a result which may not surprise since the creation/annihilation modes of a scalar field on a fixed de Sitter background were Hankel functions, see §4.2,

$$\Phi_{(\pm, \vec{k})} = \gamma \frac{\pm i 2^\nu \pi^{\frac{d}{2}+1}}{\Gamma(d-\Delta)} k^{-\nu} (-\eta)^{\frac{d}{2}} H_\nu^\pm(-k\eta) \mathcal{O}_{\vec{k}}. \quad (7.13)$$

We can therefore write, for each α

$$\Phi_{(\Delta, \vec{k})} \equiv \Phi_{(+, \vec{k})} - \Phi_{(-, \vec{k})} = \tilde{\gamma} \varphi_{(\Delta, \vec{k})}(\eta) \mathcal{O}_{\vec{k}}, \quad \tilde{\gamma} \equiv \frac{2^{\nu+\frac{1}{2}} i \pi^{\frac{d+1}{2}} \gamma}{\Gamma(d-\Delta)}, \quad (7.14)$$

where $\varphi_{(\Delta, \vec{k})}(\eta)$ is the function defined in (4.20).

7.1.3 Global dS

This result can be extended consistently to global Sitter space. The dual CFT lives on the sphere S^d , which is related to \mathbb{R}^d by the stereographic projection. More precisely, the latitude ψ is given in terms of the radial coordinate $y = \tan(\frac{\psi}{2})$, while the angles ω^i on the plane map directly to the azimuthal angles on the sphere. Therefore, scalar primary operators on the sphere are related to their flat space equivalents as

$$\mathcal{O}(\tan \frac{\psi}{2}, \omega^i) = \left(2 \cos \frac{\psi}{2}\right)^\Delta \mathcal{O}_{S^d}(\psi, \omega^i). \quad (7.15)$$

In the bulk, the coordinate transformation is given by (4.9). Substituting all of this into (7.7),

$$\begin{aligned} \Phi(T, \Omega) &\propto \sum_\alpha \gamma_\alpha^\pm \lim_{T' \rightarrow \frac{\pi}{2}} \int d\psi' d^{d-1}\omega' \frac{(\tan \frac{\psi'}{2})^{d-1}}{2 \cos^2 \frac{\psi'}{2}} \left(\frac{\cos T'}{\sin T' + \cos \psi'} \right)^{\Delta-d} \\ &\quad P(T, \Omega; T', \psi', \omega')^{\Delta-d} \left(2 \cos^2 \frac{\psi'}{2} \right)^\Delta \mathcal{O}_{S^d}^\alpha(\psi', \omega') \\ &= \frac{1}{2} \sum_\alpha \gamma_\alpha^\pm \int d^d\Omega' \left(\frac{\Omega \cdot \Omega' - \sin T \pm i\delta}{\cos T} \right)^{\Delta-d} \mathcal{O}_{S^d}^\alpha(\Omega'), \end{aligned} \quad (7.16)$$

where we have used that $d\psi' d^{d-1}\omega' \sin^{d-1}\psi' = d^d\Omega'$ and reintroduced the $i\delta$ -prescription explicitly.

Some remarks are in order, to clarify this construction in global de Sitter space. First, no point (T, Ω) is uniquely invariant under a set of de Sitter isometries. In global de Sitter, there is always an antipodal point, $(-T, -\Omega)$, where $-\Omega$ denotes the spatially antipodal point on the sphere S^d . It is invariant under exactly the same isometries. In principle, it is not clear whether the CFT operators (7.16) describe local operators at either one of these points, or both at the same time. No two spacetime antipodal points are part of the same planar patch, so this subtlety does not arise there. The combinations G_{in} and G_{out} seem to suggest a distinction, because they allow to write (7.16) using only CFT operators within (or on) the light-cone of each one of the antipodal points. It would be interesting to investigate whether interactions between the fields or gravitational backreaction make it possible to distinguish between pairs of bulk local operators at antipodal points. However this falls outside of the scope of this thesis.

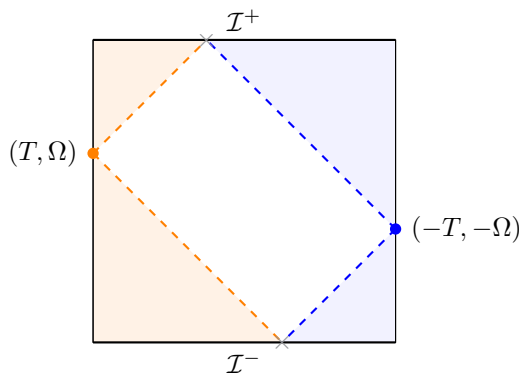


Figure 7.3: Two antipodal points on the Penrose diagram, with the associated future and past light-cones.

Another difference between the planar patch and global de Sitter is that the latter has two conformal boundaries, \mathcal{I}^+ in the asymptotic future and \mathcal{I}^- in the infinite past. It is possible to associate a CFT with each of these independently. The bulk picture provides pairings between the two: one by the bulk antipodal transformation (depicted in Figure 7.3) which maps one conformal boundary onto the other, and another map between the CFTs by bulk time evolution. This point of view was already advocated in [19]. Bulk local operators may clarify this map, at least close to the vacuum state, since they can be defined using operators of either one of the CFTs. Again, it would be interesting to

take interactions and backreaction into account, but we will not pursue this in this thesis.

7.2 The fermionic vector model

The model presented in §6.2 contains a local scalar primary operator of weight $\Delta = 1$ in $d = 3$. This is a special case, since the combinations (7.5) coincide and vanish both inside and outside the intersection of the light-cone with the conformal boundary. Thus, even though both G_{\pm} have support on the full conformal boundary, the only contribution to G_{in} comes from the singularity at $|y - x| = -\eta$.

In this section, we will refine the expressions (7.6) by performing the y -integral explicitly. We then proceed to calculate the two-point function, both in position space and in momentum space. And observe that we fail to reproduce the bulk two-point functions. In the final part of this section, we use the so-called “shadow operator” [201, 202, 203], to obtain more general two-point functions. We will find that it is still not possible to obtain the bulk two-point functions.

7.2.1 Conformally coupled scalar operator

For $\Delta = 1$ both the blue and orange branch cut in 7.2 have disappeared. Therefore, the y -integral for $\Phi_{\text{in}} = \Phi_{\text{out}}$ reduces to a closed contour around the pole. It is convenient to shift $\vec{y} \rightarrow \vec{y} + \vec{x}$ and define a unit three-vector $\vec{\omega}$ so that

$$\begin{aligned} \Phi_{\text{in}}^{(1)}(\eta, \vec{x}) &\propto \oint dy \frac{(-\eta)^2 y^2}{(\eta^2 - y^2)^2} \int_{S^2} d^2\omega \mathcal{O}^{(1)}(\vec{x} + y\vec{\omega}) \\ &= \frac{1}{2} \oint dy \frac{(-\eta)^2}{\eta^2 - y^2} \int_{S^2} d^2\omega (1 + y\partial_y) \mathcal{O}^{(1)}(\vec{x} + y\vec{\omega}) \\ &= \frac{i\pi}{2} \eta(1 + \eta\partial_{\eta}) \int_{S^2} d^2\omega \mathcal{O}^{(1)}(\vec{x} - \eta\vec{\omega}) , \end{aligned} \quad (7.17)$$

where we have used the identity $\partial_y(\eta^2 - y^2)^{-1} = 2y/(\eta^2 - y^2)^2$ and performing integration by parts. The closed integral in the first line is along a contour around the point $y = -\eta$. The same conclusion can be reached from (7.5)-(7.6) without contour deformation by distributional identity

$$\frac{1}{(1 - x \pm i\delta)^2} = \mathcal{P} \frac{1}{(1 - x)^2} \pm i\pi\delta'(1 - x) . \quad (7.18)$$

7.2.2 Two-point function

Using this result, we can investigate whether the bulk expectation values of a conformally coupled scalar on a fixed de Sitter background can be reproduced as correlation functions in the AHS model. For example, let us consider the timelike separated two-point function of $\Phi^{(1)}$ in the planar patch. We can choose the origin to coincide with the position of one of the scalar operators. Furthermore, since the second point is timelike separated from the first, we can perform a boost to bring it to the (spatial) origin as well. We are therefore interested in the following two-point correlation function

$$\langle \Phi^{(1)}(\eta, \vec{0}) \Phi^{(1)}(\eta', \vec{0}) \rangle, \quad (7.19)$$

where the CFT operator $\Phi^{(1)}$ is given holographically by (7.17). The bulk two-point function is therefore proportional to

$$\eta\eta'(1+\eta\partial_\eta)(1+\eta'\partial_{\eta'}) \int d^2\omega d^2\omega' \langle \mathcal{O}^{(1)}(-\eta\vec{\omega}) \mathcal{O}^{(1)}(-\eta'\vec{\omega}') \rangle. \quad (7.20)$$

The integrand is proportional to the CFT two-point function $c/|\eta\vec{\omega} - \eta'\vec{\omega}'|^2$. It is invariant under simultaneous rotations of ω and ω' , so we can fix one of them on the “north pole” of S^2 and multiply by $4\pi^2$. Furthermore, the remaining integrand is still invariant under rotations that leave the north pole invariant. The result therefore only depends on the angle θ between ω and ω' ,

$$\begin{aligned} & 8\pi^3 \eta\eta'(1+\eta\partial_\eta)(1+\eta'\partial_{\eta'}) \int_0^\pi d\theta \frac{c \sin \theta}{\eta^2 + \eta'^2 - 2\eta\eta' \cos \theta} \\ &= -8\pi^3 c \left(\frac{\eta\eta'}{(\eta - \eta')^2} + \frac{\eta\eta'}{(\eta + \eta')^2} \right), \end{aligned} \quad (7.21)$$

where c is the central charge of $\mathcal{O}^{(1)}$. The result does not equal a Euclidean propagator in the bulk, but instead is approximately equal to the Euclidean-to-Neumann propagator (4.41) in the bulk, albeit without the $i\epsilon$ prescription. Notice that this $i\epsilon$ is not related to the $i\delta$ -prescription used earlier, since we have taken the limit $\delta \rightarrow 0$ (7.17). Furthermore, it is tempting to add an $i\epsilon$ prescription to (7.21) by hand. However, that would require modifying the operators $\Phi^{(1)}$ depending on their position in the correlation function, or adding extra structure to the CFT correlation functions on top of their definition as a Euclidean path integral.

The same calculation can be done in momentum space, using the operator (7.14),

$$\Phi_{(1,\vec{k})}(\eta) = -\tilde{\gamma}\eta \cos(-k\eta) \mathcal{O}_{\vec{k}}^{(1)}. \quad (7.22)$$

This gives rise to the momentum space two-point function

$$\langle \Phi_{(1,\vec{k})}(\eta) \Phi_{(1,\vec{k}')}(\eta') \rangle = 2\pi^2 c \tilde{\gamma}^2 \frac{\eta \eta'}{k} \cos(-k\eta) \cos(-k'\eta') \delta^{(3)}(\vec{k} + \vec{k}') , \quad (7.23)$$

which does not equal the momentum space Euclidean-to-Neumann propagator (4.39). In fact, since it is symmetric under exchange of η and η' , it does not give rise to the characteristic momentum space commutator (4.34). This seems to be a direct result of the fact that all operators in the CFT path integral commute, and hence all correlation functions commutators will vanish.

We conclude therefore that the CFT cannot reproduce the bulk correlation functions.

7.2.3 The shadow operator

There is one additional possibility we want to investigate, namely the shadow operator [201, 202]. For any local primary operator \mathcal{O} with weight Δ in the CFT, is possible to obtain a nonlocal operator $\tilde{\mathcal{O}}$ which satisfies the condition

$$\langle \tilde{\mathcal{O}}(\vec{x}) \tilde{\mathcal{O}}(\vec{y}) \rangle = \delta^{(d)}(\vec{x} - \vec{y}) , \quad (7.24)$$

Indeed, we can define the following *shadow operator* in momentum space

$$\tilde{\mathcal{O}}_{\vec{k}} \equiv k^{d-2\Delta} \mathcal{O}_{\vec{k}} , \quad \langle \tilde{\mathcal{O}}_{\vec{k}} \tilde{\mathcal{O}}_{\vec{k}'} \rangle = \delta^{(d)}(\vec{k} - \vec{k}') , \quad (7.25)$$

which indeed leads to (7.24) upon Fourier transformation. The expression of the shadow operator in position space is given by the convolution of the original local primary with its two-point function,

$$\tilde{\mathcal{O}}(\vec{x}) \propto \int d^d y |\vec{x} - \vec{y}|^{2(\Delta-d)} \mathcal{O}(\vec{y}) . \quad (7.26)$$

Whereas this operator is indeed nonlocal, it transforms under the conformal symmetries exactly as a local primary operator of weight $\tilde{\Delta} = d - \Delta$. We can therefore define a bulk-local operator in terms of the shadow field using the formula (7.6) with $\mathcal{O} \rightarrow \tilde{\mathcal{O}}$. The resulting bulk-local operator represents a field with the same mass.

For the fermionic vector model, the shadow transforms as a local primary with weight 2, and therefore defines a bulk-local operator as $\Phi^{(2)}$ as follows,

$$\begin{aligned} \Phi_{\text{in}}^{(2)}(\eta, \vec{x}) &\propto \oint dy \frac{-\eta y^2}{\eta^2 - y^2} \int_{S^2} d^2 \omega \tilde{\mathcal{O}}(\vec{x} + y \vec{\omega}) \\ &= -i\pi \eta^2 \int_{S^2} d^2 \omega \tilde{\mathcal{O}}(\vec{x} - \eta \vec{\omega}) . \end{aligned} \quad (7.27)$$

The closed integral in the first line is again along a contour around the point $y = -\eta$. The same result can be obtained from the equivalence, in the distributional sense, of

$$\frac{1}{1-x \pm i\delta} = \mathcal{P} \frac{1}{1-x} \mp i\pi\delta(1-x) . \quad (7.28)$$

The timelike separated two-point function of $\Phi^{(2)}$ in the planar patch can be calculated as before,

$$\begin{aligned} \langle \Phi^{(2)}(\eta, \vec{0}) \Phi^{(2)}(\eta', \vec{0}) \rangle &\propto \eta^2 \eta'^2 \int d^2\omega d^2\omega' \langle \tilde{\mathcal{O}}(-\eta\vec{\omega}) \tilde{\mathcal{O}}(-\eta'\vec{\omega}') \rangle_{\text{CFT}} \\ &= 4\pi^3 c \left(\frac{\eta\eta'}{(\eta-\eta')^2} - \frac{\eta\eta'}{(\eta+\eta')^2} \right) . \end{aligned} \quad (7.29)$$

This result is approximately the Euclidean-to-Dirichlet two-point function (4.40), except for the missing $i\epsilon$ prescription.

In momentum space, formula (7.14) with $\Delta \rightarrow \tilde{\Delta}$ and $\nu \rightarrow -\nu$ gives

$$\Phi_{(2,\vec{k})} = -\tilde{\gamma} \frac{\eta}{k} \sin(-k\eta) \tilde{\mathcal{O}}_{\vec{k}}^{(2)} \quad (7.30)$$

The corresponding two-point function is

$$\langle \Phi_{(2,\vec{k})}(\eta) \Phi_{(2,\vec{k}')}(\eta') \rangle = 2\pi^2 c \tilde{\gamma}^2 \frac{\eta\eta'}{k} \sin(-k\eta) \sin(-k'\eta') \delta^{(3)}(\vec{k} + \vec{k}') , \quad (7.31)$$

which again is symmetric under $\eta \leftrightarrow \eta'$ does not equal the Euclidean-to-Dirichlet two-point function in the bulk.

More generally, we could define a bulk-local operator in the CFT as a linear combination $\Phi \equiv A\Phi^{(1)} + B\Phi^{(2)}$. Using the previous results and the characteristic two-point function (7.24) of an operator with its shadow, the corresponding two-point function in momentum space yields

$$\begin{aligned} \langle \Phi_{\vec{k}}(\eta) \Phi_{\vec{k}'}(\eta') \rangle &\propto 2\pi^2 c \tilde{\gamma}^2 \frac{\eta\eta'}{k} [A \cos(-k\eta) + B \sin(-k\eta)] \\ &\cdot [A \cos(-k\eta') + B \sin(-k\eta')] \delta^{(3)}(\vec{k} + \vec{k}') . \end{aligned} \quad (7.32)$$

This is again symmetric in $\eta \leftrightarrow \eta'$. For no values of A and B does this reproduce the Euclidean propagator (4.29). The structure of bulk correlation functions is different because of the presence of creation and annihilation operators. Unlike the CFT correlation functions, which are given by a Euclidean path integral, the commutation relations of bulk field operators makes their correlation functions sensitive to their ordering. We conclude the CFT does not have this structure, and therefore cannot account for the correlation functions in the bulk.

Chapter 8

Fermionic Hilbert space for de Sitter

The [AHS](#) model allows for very explicit calculations and consistency checks. All gauge-invariant primary operators in the [CFT](#) are known explicitly and their correlation functions can be calculated straightforwardly. This allows for direct calculations which form important consistency checks of the [dS/CFT](#) framework. The problems observed in [\[27\]](#) along with the results of the previous section are challenges for the proposal. As we discussed in [§4.5](#), the [dS/CFT](#) proposal does not clarify what the Hilbert space is of quantum gravity in de Sitter space. If it was known explicitly, it would clarify the question of normalizability of the bulk wave function and specify the operator content of quantum gravity.

In this chapter, we will take a step back from the [dS/CFT](#) proposal as it is formulated in the literature and analyze an explicit Hilbert space from which, in [Chapter 10](#), we will attempt to construct the Hilbert space of quantum Vasiliev gravity in de Sitter space. Our proposal is that it can be constructed as the singlet sector of a Hilbert space with fermionic operators that transform in the vector representation of the $U(N)$ gauge group. The fermionic operators satisfy canonical anti-commutation relations with their Hermitian conjugates.

This ansatz shares some of its structure and symmetries with the [AHS](#) model but, it is important to note, this model is logically distinct from [dS/CFT](#).

In this chapter, we will find that the singlet sector of the fermionic Hilbert space can be associated to a classical bosonic theory of which the phase space is a compact Kähler manifold, described by [Berezin](#) in [\[204\]](#). In the first section,

we will start from a simplified model, after which we review Berezin's results that characterize the gauge-invariant sector of the full Hilbert space.

Further analysis follows in the next chapters. In Chapter 9, we apply the formalism of Berezin coherent states to the quantum mechanics of Grassmann matrices. We supply the system with a gauge invariant Hamiltonian and analyze its dynamics using many of the techniques detailed in this chapter. We comment on potential holographic applications of this model towards gravitational theories with finite entropy.

In Chapter 10, we attempt to interpret the Berezin Hilbert space as the Hilbert space of fields in a dynamic asymptotically de Sitter Universe, namely as the UV completion of Vasiliev gravity.

8.1 A fermionic Hilbert space

The Hilbert space we will consider in the remainder of this thesis contains a vacuum state $|0\rangle$ annihilated by fermionic operators $a^A(x)$, with $A = 1, \dots, N$ and $x \in \mathbb{R}^3$, which transform in the fundamental representation of $U(N)$. Furthermore, there is another set of fermionic annihilation operators $b_A(x)$ which transform in the anti-fundamental representation of $U(N)$. Consider the corresponding creation operators $a_A^\dagger(x)$ and $b^{A\dagger}(x)$ which satisfy the anti-commutation relations

$$\{a^A(x), a_B(y)^\dagger\} = \delta_B^A g(x, y) = \{b_B(x), b^{A\dagger}(y)\} . \quad (8.1)$$

while other anti-commutators vanish. Consistency requires that $a_A^\dagger(x)$ and $b^{A\dagger}(x)$ transform in the anti-fundamental and in the fundamental representations of $U(N)$, respectively. These anti-commutation relations give rise to a Hilbert space with a positive definite inner product defined by identifying $\langle 0| \equiv (|0\rangle)^\dagger$, i.e. employing Dirac's bra-ket notation.

The bilocal function g is strongly constrained by symmetry. Assuming translational and rotational symmetry makes it a function of $|x - y|$. If we furthermore assume a scaling symmetry, as in a CFT or on a late-time equal-time slice of de Sitter, it can essentially only be proportional to a certain power of $|x - y|$ given by the scaling dimension of a^A and b_A . In our case, we will be interested in $g(x, y)$ equal to the two-point function of a free scalar in $d = 3$,

$$g(x, y) = \frac{c}{|x - y|} , \quad (8.2)$$

where c is a constant to be determined. It has to be real, since the \dagger -operation acts on numbers as complex conjugation.

Somewhat similar to how a spatial Fourier transform in the planar patch of de Sitter (4.14) diagonalizes the spatial Laplacian, it is also possible to perform a change of basis to make the anti-commutation relations diagonal,

$$\{a_p^A, a_{qB}^\dagger\} = \delta_B^A \delta_{pq} = \{b_{pB}, b_q^{A\dagger}\} . \quad (8.3)$$

For g as in (8.2), these operators are defined by

$$a^A(x) = \sqrt{\frac{c}{2\pi^2}} \int d^3p \frac{e^{ip \cdot x}}{|p|} a_p^A , \quad b_A(x) = \frac{\sqrt{c}}{\sqrt{2\pi}} \int d^3p \frac{e^{-ip \cdot x}}{|p|} b_{pA} , \quad (8.4)$$

and their \dagger -conjugates, labeled by “indices” $p, q \in \mathbb{R}^3$. Notice that this is not the usual Fourier transform. Instead, the operators are rescaled by a power of p , which accounts for the two-point function g . A priori, the labels p and q take on continuous values, but a convenient regularization of the theory is to consider only a discrete set of them and take the continuum limit in the end. We will often find it convenient to furthermore restrict their number K to be finite. This way, we truncate to a 2^{2KN} -dimensional part of the Hilbert spaces and will be able to recast the calculations in the language of matrices.

We will restrict the physical Hilbert space to be the $U(N)$ -invariant part of the full 2^{2KN} -dimensional Hilbert space, i.e. the $U(N)$ is a gauge symmetry of the system. In the analysis of Chapter 9, we will interpret the $U(K)$ group as an internal symmetry of a quantum mechanical particle. In the context of the Vasiliev model, we will interpret the p, q indices as in (8.4) and take the $K \rightarrow \infty$ limit to describe arbitrary continuous field profiles.

Before analyzing this full model, we will first consider the simplified case $K = 1$ and introduce many techniques employed in the analysis of the full model in §8.3.

8.2 A toy model: $K = 1$

In the case $K = 1$, the anticommutation relations (8.1) reduce to

$$\{a^A, a_B^\dagger\} = \delta_B^A = \{b_B, b^{A\dagger}\} , \quad (8.5)$$

while other anti-commutators vanish. The full (non- $U(N)$ invariant) Hilbert space is 2^{2N} dimensional. The gauge-invariant part of the Hilbert space can be constructed explicitly using the operators

$$\hat{J}_+ \equiv a_A^\dagger b^{A\dagger} , \quad \hat{J}_- \equiv b_A a^A , \quad \hat{J}_3 \equiv \frac{1}{2}(a_A^\dagger a^A - b_A b^{A\dagger}) , \quad (8.6)$$

where the $U(N)$ -index A is summed over. The notation \hat{J} is used here because these bilinear operators satisfy the commutation relations of the $SO(3)$ raising and lowering operators,

$$[\hat{J}_-, \hat{J}_+] = -2\hat{J}_3, \quad [\hat{J}_3, \hat{J}_\pm] = \pm\hat{J}_\pm, \quad (8.7)$$

which can be calculated directly from the fermionic anticommutators (8.6). The eigenvalue of the vacuum under the \hat{J}_3 operator is $-N/2$. By acting on it recursively with \hat{J}_+ , one can build an orthonormal basis of the gauge-invariant Hilbert space,

$$|n\rangle = \sqrt{\frac{(N-n)!}{N!n!}} (\hat{J}^+)^n |0\rangle, \quad (8.8)$$

for $0 \leq n \leq N$. The gauge-invariant part of this fermionic Hilbert space is that of a quantum mechanical spinning particle with fixed total angular momentum $|\vec{J}| = N/2$.

8.2.1 Coherent states

Coherent states were first introduced as a family of minimal uncertainty states of the quantum harmonic oscillator [205], where the creation and annihilation operator satisfy the Heisenberg algebra. The algebra (8.7) is different, but it is still possible to define *Bloch coherent states*¹ as

$$|\bar{z}\rangle \equiv e^{\bar{z}\hat{J}_+} |0\rangle. \quad (8.9)$$

We will denote these states as a “round kets” because they are not unit normalized vectors if $z \neq 0$. By this definition, these coherent states are $U(N)$ -invariant. In fact, they span the full $U(N)$ -invariant Hilbert space, in a way we will make more precise below.

In order to calculate the overlap between two Bloch coherent states, observe that the norm of any state created by a function $f(a^\dagger)$ of the creation operators can be expressed as a *Berezin integral* [194] (or, equivalently, a *Grassmann integral*),

$$\langle 0 | \bar{f}(a) f(a^\dagger) | 0 \rangle = \int d\tilde{\alpha} d\alpha e^{\alpha^A \tilde{\alpha}_A} \bar{f}(\alpha) f(\tilde{\alpha}), \quad (8.10)$$

where α^A and $\tilde{\alpha}_A$ are independent Grassmann numbers. The order of them in the measure $d\alpha$ does not matter as long as it is chosen the same way in $d\tilde{\alpha}$.

¹ These states are called Bloch coherent states because, as we will see below, they parametrize a classical phase space called the Bloch sphere.

The Berezin integral selects by definition the terms in the integrand for which each fermionic variable appears exactly once. One can see by expanding the exponential that this will pick up all terms in $\bar{f}f$ which have an α^1 for every $\tilde{\alpha}_1$, etc. Therefore it is equivalent to the inner product on the left-hand side of (8.10).

The inner product of two Bloch coherent states made from fermionic operators satisfying (8.5) can thus be written as

$$\begin{aligned} (y|\bar{z}) &= \int d\tilde{\alpha} d\alpha d\tilde{\beta} d\beta e^{\alpha^A \tilde{\alpha}_A + \beta_A \tilde{\beta}^A} e^{y\beta_A \alpha^A} e^{\bar{z}\tilde{\alpha}_A \tilde{\beta}^A} \\ &= \int d\tilde{\alpha} d\alpha d\tilde{\beta} d\beta \exp \left[(\beta \quad \tilde{\alpha}) \begin{pmatrix} y & 1 \\ -1 & \bar{z} \end{pmatrix} \begin{pmatrix} \alpha \\ \tilde{\beta} \end{pmatrix} \right], \end{aligned} \quad (8.11)$$

where the sum over A has been suppressed in the last line. The integrand factorizes into a product of N Gaussians, the Berezin integral of which is the determinant of the matrix in the exponent. Hence the result

$$(y|\bar{z}) = \det \begin{pmatrix} y & 1 \\ -1 & \bar{z} \end{pmatrix}^N = (1 + \bar{z}y)^N. \quad (8.12)$$

We can therefore define normalized coherent states $|\bar{z}\rangle \equiv |\bar{z}\rangle / (1 + z\bar{z})^{N/2}$.

8.2.2 Holomorphic wave functions and symbols

It is clear from (8.12) that the Bloch coherent states are not an orthogonal set of states. Instead, they span the full Hilbert space in an overcomplete manner. This allows us to reformulate operators and states in terms of functions of (z, \bar{z}) and extract calculational rules for these objects, as we will now explain.

For any state $|\psi\rangle$, one can define the *holomorphic wave function*

$$\psi(z) \equiv (z|\psi), \quad (8.13)$$

which contains the full information about $|\psi\rangle$. For example, the holomorphic wave function of the vacuum is 1, whereas the holomorphic wave functions corresponding to any of the \hat{J}_3 -eigenstates $|m\rangle$ is the monomial of degree m in z . Since these form a basis of the gauge-invariant Hilbert space, another characterization of the latter is the set of all polynomials in z up to degree N . The holomorphic wave function of a Bloch coherent state was already given in (8.12). The operators \hat{J} act as differential operators on holomorphic wave functions: $(z|\hat{J}_i|\psi) = \mathcal{D}_{(J_i)}\psi(z)$ with

$$\mathcal{D}_{(J_-)} = \partial_z, \quad \mathcal{D}_{(J_+)} = Nz - z^2\partial_z, \quad \mathcal{D}_{(J_3)} = z\partial_z - N/2. \quad (8.14)$$

Similarly, for any operator \hat{A} , one can define the *symbol*

$$A(z, \bar{z}) \equiv \langle z | \hat{A} | \bar{z} \rangle . \quad (8.15)$$

Since the full set of Bloch coherent states span the full Hilbert space, the matrix element of an operator between two coherent states describes the full behavior of that operator. An example of a symbol is given by the expectation value of the angular momentum operator,

$$\vec{J}(z, \bar{z}) \equiv \frac{(z | \hat{\vec{J}} | \bar{z})}{(z | \bar{z})} = \frac{N}{2} \left(\frac{z + \bar{z}}{1 + \bar{z}z}, \frac{1}{i} \frac{z - \bar{z}}{1 + \bar{z}z}, -\frac{1 - \bar{z}z}{1 + \bar{z}z} \right) . \quad (8.16)$$

This is the stereographic parameterization of the vector \vec{J} on the two-sphere of radius $N/2$ in complex coordinates. Furthermore, the inner product of two normalized Bloch coherent states is

$$|\langle y | \bar{z} \rangle|^2 = \frac{(1 - \bar{z}y)^N}{(1 - \bar{y}y)^{\frac{N}{2}} (\frac{N}{2} - \bar{z}z)^{\frac{N}{2}}} = \left(\cos \frac{\theta}{2} \right)^{2N} , \quad (8.17)$$

where θ is the angle between the \vec{J} -vectors associated to y and to z , respectively. In the large N limit, this becomes sharply peaked around these vectors being aligned: when $y \approx z$ in the complex plane, we have $|\langle y | \bar{z} \rangle| \approx e^{-N\theta^2/8}$. This clarifies the physical meaning of the coherent states as highly localized states on the two-sphere of angular momenta \vec{J} with $|\vec{J}| = N/2$.

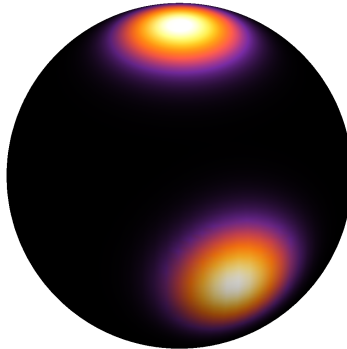


Figure 8.1: This figure represents the overlap of the Bloch coherent state $|z\rangle$ – where z is the stereographic coordinate on this two-sphere – with a linear combination of two other coherent states for $N = 50$. Lighter colors represent higher overlap.

8.2.3 Rotations

Since this fermionic Hilbert space is that of spinning particle, the vacuum $|0\rangle$ is only one of a class of “lowest \hat{J}^3 eigenstates” for different orientations of the two-sphere. In other words, a two (real) parameter family of states with minimal uncertainty can be obtained from the vacuum by $SO(3)$ rotations.

The fermionic formulation has a similar ambiguity: there are other operators that satisfy the anti-commutation relations (8.5) and thus furnish an equivalent basis of operators for the Hilbert space,

$$\tilde{a}^A \equiv \bar{y}a^A + \bar{x}b^{A\dagger}, \quad \tilde{b}_A = va_A^\dagger + wb_A, \quad (8.18)$$

and their \dagger -conjugates. The coefficients v and w can be fixed in terms of \bar{x} and \bar{y} up to a common phase by the analog of (8.5) for \tilde{b} , and by the requirement that \tilde{a} and \tilde{b} anticommute. Let us therefore denote by $|\bar{x}, \bar{y}\rangle$ the corresponding “vacuum” state, i.e. the normalized state annihilated by the \tilde{a} and \tilde{b} operators. It corresponds to a state with the same properties as the original vacuum. It is not influenced by the undetermined common phase shift in v and w , nor is it influenced by a similar phase in \bar{x} and \bar{y} .

One more constraint can be derived from the requirement that (8.5) holds for the \tilde{a} : $\bar{y}y + \bar{x}x = 1$. It is preserved by a $U(2)$ group which acts linearly on \bar{x} and \bar{y} , but as mentioned before, the $U(1)$ subgroup of common phase rotations does not change the vacuum $|\bar{x}, \bar{y}\rangle$. The remaining set of inequivalent vacua is the Lie group $SO(3) = SU(2)/\mathbb{Z}_2$, where the \mathbb{Z}_2 operation changes the sign of \bar{x} and \bar{y} simultaneously. It acts linearly on \bar{x} and \bar{y} ,

$$\begin{cases} \bar{x} & \rightarrow a\bar{x} + b\bar{y} , \\ \bar{y} & \rightarrow c\bar{y} + d\bar{x} , \end{cases} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)/\mathbb{Z}_2 . \quad (8.19)$$

On other words, the manifold of inequivalent vacua $|\bar{x}, \bar{y}\rangle$ is two-sphere. Their explicit expression can be found by solving the equation $(\bar{y}a^A + \bar{y}b^{A\dagger})|\bar{x}, \bar{y}\rangle = 0$,

$$|\bar{x}, \bar{y}\rangle \propto e^{-a_A^\dagger b^{A\dagger} \bar{x}/\bar{y}} |0\rangle , \quad (8.20)$$

i.e. they are Bloch coherent states with $\bar{z} = \bar{x}/\bar{y}$ and the S^2 of equivalent vacua corresponds to the distinct orientations of the spinning particle in the Bosonic picture. On the original coordinate z used to parameterize Bloch coherent states, this rotation group therefore acts as

$$z \rightarrow \frac{az + b}{cz + d} . \quad (8.21)$$

8.2.4 Decomposition of unity

Using the fact that Bloch coherent states transform under $SO(3)$ rotations as in (8.21), we can express the (over)completeness of Bloch coherent states as a decomposition of the unit operator on the Hilbert space,

$$\mathbb{1} = \int [d^2 z] |\bar{z}\rangle \langle z|, \quad [d^2 z] = \frac{N+1}{\pi} \frac{dz d\bar{z}}{(1+\bar{z}z)^2}. \quad (8.22)$$

This is a result of Schur's lemma: since the spin $\frac{N}{2}$ Hilbert space transforms in an irreducible representation of $SO(3)$ rotations and since the integral is invariant under the transformations (8.21), it must be proportional to the identity operator on the Hilbert space. The normalization is fixed by the requirement that $1 = \langle 0|\mathbb{1}|0\rangle$. Note that the integration measure is the metric on the two-sphere in stereographic coordinates.

The decomposition of unity is a very useful equation because it can be used to express Hilbert space calculations purely in terms of symbols and holomorphic wave functions. For example, the inner product of two states can be written in terms of their holomorphic wave functions using (8.22),

$$\langle \psi|\phi\rangle = \int [d^2 z] \frac{\bar{\psi}(z)\phi(z)}{(1+\bar{z}z)^N}. \quad (8.23)$$

In the same vein, the symbol corresponding to the product of two operators can be expressed as the *star product* of the symbols,

$$\begin{aligned} (A \star B)(z, \bar{z}) &\equiv \langle z|\hat{A}\hat{B}|\bar{z}\rangle = \int [d^2 y] \langle z|\hat{A}|\bar{y}\rangle \langle y|\hat{B}|\bar{z}\rangle \\ &= \int [d^2 y] |\langle y|z\rangle|^2 A(z, \bar{y}) B(y, \bar{z}), \end{aligned} \quad (8.24)$$

where we have used the analytic continuation of the symbols

$$A(y, \bar{z}) = \frac{(y|\hat{A}|\bar{z})}{(1+\bar{z}y)^N} \neq \frac{(y|\hat{A}|\bar{z})}{(1+\bar{y}y)^{\frac{N}{2}}(1+\bar{z}z)^{\frac{N}{2}}}. \quad (8.25)$$

This is the analytic continuation from $\mathbb{C} \rightarrow \mathbb{C}^2$ where the two variables are no longer conjugate to each other. It does retain holomorphicity in y and anti-holomorphicity in z . The factor $|\langle y|\bar{z}\rangle|^2$ in the integral expression of the star product results from the non-orthogonality of coherent states. However, as explained after (8.17), the overlap between coherent states is exponentially suppressed. For operators \hat{A} and \hat{B} that are not sharply peaked themselves in the classical limit, the star product thus reduces to the normal product $(A \star B)(z, \bar{z}) \approx A(z, \bar{z})B(z, \bar{z})$.

8.2.5 Classical approximation

The previous argument can be made more precise by expanding the integrand in (8.24) around $y \approx z$. Consider first the star product evaluated at the origin of the complex z -plane,

$$(A \star B)(0, 0) = \frac{N+1}{\pi} \int d^2 y \frac{A(0, \bar{y})B(y, 0)}{(1 + \bar{y}y)^{N+2}}. \quad (8.26)$$

Replacing $A(0, \bar{y})B(y, 0)$ by a general function $f(y, \bar{y})$, we can evaluate the integral by replacing f by its expansion

$$\begin{aligned} & \int d^2 y \frac{f(y, \bar{y})}{(1 + \bar{y}y)^{N+2}} \\ &= \int d^2 y \frac{1}{(1 + \bar{y}y)^{N+2}} \sum_{n, m \geq 0} \frac{\bar{y}^m y^n}{m! n!} (\partial_{\bar{w}})^m (\partial_w)^n f(w, \bar{w})|_{w=0}, \end{aligned} \quad (8.27)$$

and calculate the resulting integral over the complex y plane exactly. It vanishes for $m \neq n$ because a nonzero phase integrates to zero. Furthermore, it only converges for $n < N + 1$. For higher-order terms, the integral does not converge: our result will only be an asymptotic series for $N \rightarrow \infty$. We have

$$\int d^2 y \frac{(\bar{y}y)^n}{(1 + \bar{y}y)^{N+2}} = 2\pi \cdot \frac{n! (N - n)!}{2(N + 1)!}, \quad (8.28)$$

For $n \ll N$, subsequent coefficients are suppressed by powers of $1/N$. For sufficiently well-behaved A and B , this still provides a useful expansion. The resulting approximation to the star product at an arbitrary point (z, \bar{z}) can be obtained by an $SO(3)$ rotation. The rotation (8.21) that maps the origin onto z , maps w onto $\frac{w+z}{1-\bar{z}w}$. Therefore,

$$(A \star B)(z, \bar{z}) \approx \sum_{n \geq 0} \frac{(N - n)!}{N! n!} (\partial_{\bar{w}} \partial_w)^n A\left(z, \frac{\bar{w} + \bar{z}}{1 - \bar{z}w}\right) B\left(\frac{w + z}{1 - \bar{z}w}, \bar{z}\right) \Big|_{w=0}. \quad (8.29)$$

The star-commutator of two operators is of particular interest. Using (8.29) to calculate the leading contribution to the symbol of the commutator of two operators, we get

$$[A, B]_\star \approx \frac{1}{N} (1 + \bar{z}z)^2 (\partial_{\bar{z}} A \partial_z B - \partial_z A \partial_{\bar{z}} B) + \mathcal{O}\left(\frac{1}{N^2}\right). \quad (8.30)$$

We see that to leading order, this commutator has the form of a (classical) Poisson bracket written in complex coordinates $z = x + ip$. This quantum theory

is therefore associated to a classical *phase space* with symplectic form

$$\omega^{(2)} = iN \frac{dz \wedge d\bar{z}}{(2 + \bar{z}z)^2} . \quad (8.31)$$

This symplectic form is related to the $SU(2)$ -invariant metric used in the decomposition of unity (8.22): they are derived from a Kähler potential

$$\mathcal{K} = 2N \log(1 + \bar{z}z) , \quad [d^2 z] \propto \partial_z \partial_{\bar{z}} \mathcal{K} dz d\bar{z} , \quad \omega^{(2)} = \frac{i}{2} \partial_z \partial_{\bar{z}} \mathcal{K} dz \wedge d\bar{z} . \quad (8.32)$$

Therefore, the phase space is a compact Kähler manifold, the two-sphere.

8.3 The general case: $K > 1$

The machinery developed in the previous section can be applied to the more general fermionic Hilbert space ansatz (8.3) to show that there can also be a phase space associated to it. This phase space was analyzed by Berezin in [194]. In this section, we will review the techniques he developed in the context of the Hilbert space (8.3).

8.3.1 Berezin coherent states

As in the $SO(3)$ toy model, we will eventually be interested only in the $U(N)$ -invariant part of the Hilbert space, i.e. we will declare the $U(N)$ symmetry a gauge symmetry and consider only invariant states and operators to be physical. The invariant combinations of the elementary fermionic operators are

$$\begin{aligned} \hat{\mathcal{J}}_{pq}^+ &= a_{pA}^\dagger b_q^{A\dagger} , & \hat{\mathcal{J}}_{pq}^- &= b_{pA} a_q^A , \\ \hat{\mathcal{F}}_{pq} &= a_{pA}^\dagger a_q^A , & \hat{\mathcal{G}}_{pq} &= b_q^{A\dagger} b_{pA} , \end{aligned} \quad (8.33)$$

where the $U(N)$ -index A is summed over. Notice the order of the p and q indices in the definition of $\hat{\mathcal{G}}$. The physical Hilbert space can be generated by acting on the vacuum with $\hat{\mathcal{J}}_{pq}^+$, which thus acts roughly as a creation operator. It is not a creation operator in the usual sense, because its commutator with the corresponding annihilation operator $(\hat{\mathcal{J}}_{pq}^+)^\dagger = \hat{\mathcal{J}}_{qp}^-$ is

$$[\hat{\mathcal{J}}_{pq}^-, \hat{\mathcal{J}}_{rs}^+] = N \delta_{rq} \delta_{ps} - \delta_{rq} \hat{\mathcal{G}}_{ps} - \delta_{ps} \hat{\mathcal{F}}_{rq} . \quad (8.34)$$

This is the analog of the corresponding $so(3)$ commutator and is consistent with the finite dimensionality of the Hilbert space: if only the first term on the

right-hand side would be present, $\hat{\mathcal{J}}^\pm$ would be proportional to the creation and annihilation operators of the Heisenberg algebra, which does not allow any finite-dimensional representations. Since $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$ are normal ordered, the correction is subleading in N for states “close to the vacuum”, i.e. states that can be obtained from the vacuum by acting n times with a $\hat{\mathcal{J}}^+$ operators, for $n \ll N$. The other non-vanishing commutators are

$$\begin{aligned} [\hat{\mathcal{F}}_{pq}, \hat{\mathcal{J}}_{rs}^+] &= \delta_{rq} \hat{\mathcal{J}}_{ps}^+, & [\hat{\mathcal{F}}_{pq}, \hat{\mathcal{J}}_{rs}^-] &= -\delta_{ps} \hat{\mathcal{J}}_{rq}^-, \\ [\hat{\mathcal{G}}_{pq}, \hat{\mathcal{J}}_{rs}^+] &= \delta_{ps} \hat{\mathcal{J}}_{rq}^+, & [\hat{\mathcal{G}}_{pq}, \hat{\mathcal{J}}_{rs}^-] &= -\delta_{rq} \hat{\mathcal{J}}_{ps}^-, \\ [\hat{\mathcal{F}}_{pq}, \hat{\mathcal{F}}_{rs}] &= \delta_{rq} \hat{\mathcal{F}}_{ps} - \delta_{ps} \hat{\mathcal{F}}_{rq}, & [\hat{\mathcal{G}}_{pq}, \hat{\mathcal{G}}_{rs}] &= \delta_{ps} \hat{\mathcal{G}}_{rq} - \delta_{rq} \hat{\mathcal{G}}_{ps}. \end{aligned} \quad (8.35)$$

The first two rows are similar to the equation $[\hat{\mathcal{J}}^3, \hat{\mathcal{J}}^\pm] = \pm \hat{\mathcal{J}}^\pm$ in for the spinning particle. The last row has no analog, since $\hat{\mathcal{J}}^3$ commutes with itself.

Gauge-invariant coherent states can be defined with these bilinear operators. In this case they are parameterized by a complex bilocal function, or in our notation a complex matrix Z_{pq} ,

$$\begin{aligned} |Z^\dagger\rangle &\equiv e^{\text{Tr}(Z^\dagger \hat{\mathcal{J}}^+)} |0\rangle = e^{Z_{pq}^* a_p^\dagger b_q^{A\dagger}} |0\rangle, \\ \langle Z| &= \langle 0| e^{\text{Tr}(Z \hat{\mathcal{J}}^-)} = \langle 0| e^{Z_{pq} b_q^A a_p^A}. \end{aligned} \quad (8.36)$$

These coherent states were first considered by Berezin [204] in the context of quantizing the Gross-Neveu model [206]. The inner product of two of these states can be calculated using Berezin integrals, as in (8.11). The derivation is completely analogous and the result is the direct extension,

$$(Y|Z^\dagger) = \det \begin{pmatrix} Y & \mathbb{1} \\ -\mathbb{1} & Z^\dagger \end{pmatrix}^N = \det(\mathbb{1} + Z^\dagger Y)^N, \quad (8.37)$$

where $\mathbb{1}$ is the $K \times K$ identity matrix. Using this result, we can define normalized coherent states $|Z^\dagger\rangle = \det(\mathbb{1} + Z^\dagger Z)^{-\frac{N}{2}} |Z^\dagger\rangle$.

8.3.2 Holomorphic wave functions and operator symbols

Berezin coherent states completely span the $U(N)$ -invariant part of the fermionic Hilbert space. As before, we can relate a holomorphic wave function to any state in that Hilbert space by calculating the inner product with a coherent bra,

$$\psi(Z) = (Z|\psi). \quad (8.38)$$

This is now a function of K^2 complex coordinates. The elementary operators in (8.33) are again represented as differential operators,

$$\begin{aligned}\mathcal{D}_{pq}^{(-)} &= \partial_{Z_{qp}} , & \mathcal{D}_{pq}^{(+)} &= N Z_{pq} - Z_{pr} Z_{sq} \partial_{Z_{sr}} , \\ \mathcal{D}_{pq}^{(\mathcal{F})} &= Z_{pr} \partial_{Z_{qr}} , & \mathcal{D}_{pq}^{(\mathcal{G})} &= Z_{sq} \partial_{Z_{sp}} .\end{aligned}\quad (8.39)$$

In order to be able to use matrix notation instead of writing p, q indices explicitly, it is convenient to introduce the notation $(\partial_Z)_{pq} \equiv \partial_{Z_{qp}} = \partial/\partial Z_{qp}$.

We will find an expression for the inner product of two states in terms of their holomorphic wave functions in the next section, where we will derive the decomposition of unity in this Hilbert space.

The space of holomorphic wave functions is not just the space of polynomials in the Z_{pq} up to some order related to N . For example, it is always possible to act N times on the vacuum with $\hat{\mathcal{J}}_{11}^+$. This state corresponds to a wave function proportional to Z_{11}^N and is annihilated not only by another $\hat{\mathcal{J}}_{11}^+$, but by $\hat{\mathcal{J}}_{1p}^+$ and $\hat{\mathcal{J}}_{p1}^+$ for any value of p . This is manifestation of the fermionic nature of the underlying creation and annihilation operators. It is possible to excite this state further with raising operators that have no p, q -index equal to 1.

The story does not end there. Consider the maximally excited state, created by acting with every single fermionic creation operator a_{pA}^\dagger or $b_p^{A\dagger}$. The corresponding wave function is not just a single term like Z_{pp}^N , but proportional to a sum over all permutations

$$\sum_{\sigma_1, \dots, \sigma_N} (-1)^{\text{sgn}(\sigma_1) + \dots + \text{sgn}(\sigma_N)} Z_{1\sigma_1(1)} \dots Z_{1\sigma_N(1)} Z_{2\sigma_1(2)} \dots Z_{K\sigma_N(K)} . \quad (8.40)$$

By virtue of its uniqueness, this state has a very symmetric holomorphic wave function. Generic highly excited states will be more complicated. In general, one should expect non-trivial effects due to underlying fermionic statistics when more than N raising operators act on the vacuum.

Berezin coherent states can be used to associate a symbol to each operator in the Hilbert space, as before. The symbol of a generic operator depends non-holomorphically on K^2 complex variables

$$A(Z, Z^\dagger) = \langle Z | \hat{A} | Z^\dagger \rangle . \quad (8.41)$$

The symbols of the operators in (8.33) can be derived conveniently by observing that $\langle Z | Z^\dagger \rangle$ is a special case of (8.38) and using the representation (8.39). For

example, the symbol of $\hat{\mathcal{J}}^-$ is

$$\begin{aligned} J_{pq}^-(Z, Z^\dagger) &= \frac{\partial_{Z_{pq}}(Z|Z^\dagger)}{(Z|Z^\dagger)} = N \frac{\partial}{\partial Z_{qp}} \text{Tr} \log(\mathbb{1} + Z^\dagger Z) \\ &= N [(\mathbb{1} + Z^\dagger Z)^{-1} Z^\dagger]_{pq} , \end{aligned} \quad (8.42)$$

where all omitted p, q indices are contracted by the rules of matrix multiplication. In particular, the inverse of $\mathbb{1} + Z^\dagger Z$ exists for any matrix Z .

Similarly, either by direct computation, through (8.39) or from complex conjugation of the result for J^- , we have

$$J^+ = NZ(\mathbb{1} + Z^\dagger Z)^{-1} , \quad (8.43)$$

as $K \times K$ matrices. Finally, the symbols of $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$ are

$$F = NZ(\mathbb{1} + Z^\dagger Z)^{-1} Z^\dagger , \quad G = N(\mathbb{1} + Z^\dagger Z)^{-1} Z^\dagger Z . \quad (8.44)$$

8.3.3 Manifold of coherent states

As before, we will be able to characterize the space of coherent states as the space of equivalent representations of the anti-commutator algebra (8.3). We can introduce equivalent operators \tilde{a}_p^A and \tilde{b}_{pA} as linear combinations of the original operators in the anti-commutation relations.

$$\tilde{a}_p^A \equiv (Y^\dagger)_p{}^q a_q^A + (X^\dagger)_p{}^q b_q^{A\dagger} , \quad \tilde{b}_{pA} = (V^\dagger)_p{}^q a_{qA}^\dagger + (W^\dagger)_p{}^q b_{qA} . \quad (8.45)$$

The matrices V^\dagger and W^\dagger can be fixed (up to a $U(K)$ subgroup which rotates the \tilde{b} into each other) in terms of Y^\dagger and X^\dagger by the requirement that the anti-commutations hold for \tilde{b} and that \tilde{b} anti-commutes with \tilde{a} . The requirement that $\{\tilde{a}_p, \tilde{a}_q^\dagger\} = \delta_{pq}$ imposes

$$X^\dagger X + Y^\dagger Y = \mathbb{1} . \quad (8.46)$$

The normalized Berezin coherent states are the “vacua” annihilated by the \tilde{a} and \tilde{b} operators:

$$(Y^\dagger a + X^\dagger b^\dagger) e^{\text{Tr}(Z^\dagger \hat{\mathcal{J}}^+)} |0\rangle = (-Y^\dagger Z^\dagger b^\dagger + X^\dagger b^\dagger) e^{\text{Tr}(Z^\dagger \hat{\mathcal{J}}^+)} |0\rangle , \quad (8.47)$$

which indeed vanishes for $Z = XY^{-1}$.

The symmetry group $U(2K)$ that acts in the fundamental representation on the a and b^\dagger operators and in the anti-fundamental on a^\dagger and b , leaves the

anti-commutation relations invariant. It is represented linearly on the X, Y coordinates,

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} AX + BY \\ CX + DY \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2K). \quad (8.48)$$

The $U(K)$ subgroup $A = D$ and $B = 0 = C$ leaves the coordinate $Z = XY^{-1}$ invariant. In other words it does not lead to a different vacuum state, since it rotates the annihilation operators \tilde{a} into each other. Recalling that the same was possible for the \tilde{b} , we conclude that the space of Berezin coherent states is a specific type of Grassmannian manifold

$$U(2K)/U(K) \times U(K). \quad (8.49)$$

This is again a Kähler manifold. It is possible to define a metric and symplectic form derived from a Kähler potential \mathcal{K} . For example, in the coordinate patch where $\det Y \neq 0$ so that $Z = XY^{-1}$ can be defined, consider

$$\mathcal{K} = \log \det(\mathbb{1} + Z^\dagger Z), \quad (8.50)$$

which is invariant under the $U(2K)$ transformations (8.48). The associated metric and symplectic structure are

$$\begin{aligned} ds^2 &= \text{Tr}[(\mathbb{1} + ZZ^\dagger)^{-1} dZ (\mathbb{1} + Z^\dagger Z)^{-1} dZ^\dagger], \\ \omega^{(2)} &= i \text{Tr}[(1 + Z^\dagger Z)^{-1} dZ \wedge (1 + ZZ^\dagger)^{-1} dZ^\dagger]. \end{aligned} \quad (8.51)$$

8.3.4 Decomposition of unity

Given how the $U(2K)$ transformations act on the coordinate Z , one can again argue by Schur's lemma that

$$\hat{\mathbb{1}} = \int [dZ] |Z^\dagger\rangle \langle Z|, \quad [dZ] \equiv c \frac{d^{K^2} Z d^{K^2} Z^\dagger}{\det(1 + Z^\dagger Z)^{2K}}, \quad (8.52)$$

where c is a normalization constant that should be fixed by the requirement $\langle 0 | \mathbb{1} | 0 \rangle$.

As in the toy model in §8.2, the decomposition of unity provides an inner product on the space of holomorphic wave functions,

$$\langle \psi_1 | \psi_2 \rangle = \int [dZ] \psi_1(Z)^* \psi_2(Z). \quad (8.53)$$

Chapter 9

Fermion matrix quantum mechanics [3]

This section is a reprint of [3] where we explore quantum mechanical theories whose fundamental degrees of freedom are rectangular matrices with Grassmann valued matrix elements. We study particular models where the low energy sector can be described in terms of a bosonic Hermitian matrix quantum mechanics. We describe the classical curved phase space that emerges in the low energy sector. The phase space lives on a compact Kähler manifold parameterized by a complex matrix, of the type discovered some time ago by Berezin. The emergence of a semiclassical bosonic matrix quantum mechanics at low energies requires that the original Grassmann matrices be in the long rectangular limit. We discuss possible holographic interpretations of such matrix models which, by construction, are endowed with a finite dimensional Hilbert space.

9.1 Introduction

Models with matrix like degrees of freedom make numerous appearances throughout physics. Applications range from the study of the spectra of heavy atoms to models of emergent geometry [207, 4, 208, 209, 210, 211]. In this paper we will concern ourselves with a particular class of quantum mechanical models whose degrees of freedom are purely fermionic rectangular matrices ψ_{Ai} , with $A = 1, \dots, M$ and $i = 1, \dots, N$. The matrices transform in the (M, N) bifundamental representation of a $U(M) \times SU(N)$ symmetry group. In a

Lagrangian description of the system, transition amplitudes can be expressed as path integrals over Grassmann valued paths ψ_{Ai} . Grassmann matrices naturally appear as the supersymmetric partners of bosonic Hermitian matrices in supersymmetric matrix quantum mechanical theories such as the low energy worldline dynamics of a stack of N D0-branes in type IIA string theory [3, 7] or the Marinari-Parisi matrix model [8]. Our interest is in quantum mechanical models consisting of *only* the Grassmann matrices.

Ordinary integrals over Grassmann matrices were studied extensively in [212, 213, 214]. There, it was shown how the problem of Grassmann matrix integrals at large N , M can be expressed as an eigenvalue problem for the composite $N \times N$ matrix $\Phi_{ij} = \sum_A \bar{\psi}_{iA} \psi_{Aj}$, which is effectively bosonic. Unlike bosonic matrices, a Grassmann valued matrix cannot be diagonalized and characterized in terms of eigenvalues. Instead, the authors were able to analyze the model by diagonalizing Φ_{ij} . Certain features of the Φ_{ij} integral, such as a contribution to the potential of the form $\text{tr} \log \Phi$, were shown to be universal and specifically related to the Grassmann nature of the original problem. Along a similar vein, emergent bosonic matrices from spin systems were considered in [215, 216]. The models of interest in our work can be viewed as multi-particle quantum mechanical models of fermions which can occupy a finite set of single particle states $|A, i, \alpha\rangle$, labeled by the matrix indices. In particular the Hilbert space is finite dimensional. Fermionic multi-particle models often arise as lattice models in condensed matter physics, where there is typically an assumption about some sort of nearest-neighbour interaction between the fermions reflecting spatial locality. In contrast, the class of models of interest in our paper have no such notion of spatial locality. They are described by actions of the form:

$$S = \int dt i \sum_{A, \alpha, i} \bar{\psi}_{iA}^{\alpha} \partial_t \psi_{Ai}^{\alpha} - \text{tr}_{N \times N} V \left(\sum_{A, \alpha, \beta} \bar{\psi}_{iA}^{\alpha} \sigma_{\alpha\beta} \psi_{Aj}^{\beta} \right). \quad (9.1)$$

The potential $V(x)$ is an $N \times N$ matrix valued function. The index α is a spinor index associated to the d -dimensional rotation group, but we will focus on the particular case of $d = 3$ and take the $\sigma_{\alpha\beta}$ to be the ordinary Pauli matrices. We will also demand that the potential $V(x)$ be $SO(3)$ invariant.¹ An example of such a model was studied in [217]. The objects we wish to understand are path integrals over $\{\bar{\psi}_{iA}^{\alpha}(t), \psi_{Ai}^{\alpha}(t)\}$ rather than simple integrals. In particular, we study to what extent the Grassmann matrix models at large N and M can be described in terms of a composite bosonic matrix degree of freedom. We then describe several features of the emergent bosonic matrix quantum mechanical

¹ Part of the reason for choosing an $SO(3)$ index is to mimic the examples of matrix quantum mechanics that appear in holography, where the matrices are labeled by a similar rotational index. We discuss this further in the outlook.

systems. We focus on the case where $V(x)$ is quartic in the Grassmann matrices, but the techniques we develop can be used more generally.

As mentioned, our models have a finite dimensional Hilbert space. In this sense they differ from many of the quantum mechanical models studied in the context of holography, such as the D0-brane quantum mechanics or $\mathcal{N} = 4$ super Yang-Mills, where the systems have an infinite space of states, even at finite N . On the other hand, several proposals have been made throughout the literature suggesting that the holographic dual of a de Sitter universe (or at least its static patch) is indeed a system with a finite dimensional Hilbert space [218, 219, 220, 221, 222, 223]. Our considerations are particularly similar, in spirit, to those of [218, 219] where the basic building blocks are also taken to be a large collection of fermionic operators. Part of our motivation is to understand to what extent systems with a finite Hilbert space can give rise to a holographic description with a dual gravitational theory in an appropriate large N type limit. In order for this to be the case, bosonic variables (such as the Hermitean matrices) should emerge from the discrete variables, at least at low energies and in an appropriate large N limit. The models studied in this work serve as toy models where this can be seen explicitly, and we can examine to what extent the bosonic effective degrees of freedom adequately capture the physics and when this description breaks down.

The first part of the paper provides a detailed study for the $N = 1$ case, in which the degrees of freedom are organized as vectors. We derive several results regarding the physics of the effective composite degree of freedom $\bar{\psi}_A^\alpha \sigma_{\alpha\beta} \psi_A^\beta$. We show to what extent the theory is described by three bosonic degrees of freedom $\mathbf{x} = (x, y, z)$ transforming as an $SO(3)$ vector. The Euclidean path integral is expressed as a path integral over \mathbf{x} and a low velocity expansion is developed at large M . We study the theories at finite temperature and note a breakdown of the bosonic description at high temperatures. We describe the structure of the emergent classical phase space for the effective bosonic theory, which is the compact Kähler manifold \mathbb{CP}^1 . Some of the results in this section have appeared in several contexts (see for example [224, 225, 226]). However, certain aspects of our treatment are novel and furthermore our treatment naturally generalizes to the matrix case. This is studied in the second part of the paper, where now the effective theory becomes that of three bosonic Hermitian $N \times N$ matrices Σ_{ij}^a , with $a \in \{x, y, z\}$. The matrix Σ_{ij}^a transforms in the adjoint of $SU(N)$ and is an $SO(3)$ vector. The matrix analogue of the emergent classical phase space is identified as a compact Kähler manifold, first introduced by Berezin [204]. The Kähler metric is parameterized by a complex $N \times N$ matrix Z_{ij} . We discuss how the Z_{ij} and Z_{ij}^\dagger relate to the description of the system in terms of the Σ_{ij}^a as well as the original Grassmann matrices. The volume of the Kähler metric computes the dimension of the Hilbert space captured by the

(quantized) classical phase space. It is shown to precisely match the dimension of the $U(M)$ invariant Hilbert space of the original Grassmann theory. We end with an outlook discussing speculative connections of our models to holography.

9.2 Vector model

In this section we discuss a quantum mechanical model in which the degrees of freedom are a vector ψ_A^α of complex Grassmann numbers, with $A = 1, \dots, M$ and $\alpha = 1, 2$ a spinor index of $SU(2)$, the double cover of the rotational group $SO(3)$. Our system has a 2^{2M} complex-dimensional Hilbert space of states. The purpose of the section is to analyze a simplified version of the matrix model studied in the next section, which however still retains some of the salient features.

We focus on an action with quartic interactions of the specific form:

$$S = \int dt i \bar{\psi}_A^\alpha \partial_t \psi_A^\alpha + g \left(\bar{\psi}_A^\alpha \sigma_{\alpha\beta}^a \psi_A^\beta \right) \left(\bar{\psi}_B^\gamma \sigma_{\gamma\delta}^a \psi_B^\delta \right) , \quad (9.2)$$

where it is understood that the A and α indices are summed over and the $\sigma_{\alpha\beta}^a = \{\sigma_{\alpha\beta}^x, \sigma_{\alpha\beta}^y, \sigma_{\alpha\beta}^z\}$ are the three Pauli matrices. The model has an $SU(2) \times U(M)$ global symmetry group. The $(\bar{\psi}_A^\alpha)$ ψ_A^α transform in the (anti-)fundamental representation of $U(M)$ and $SU(2)$.

Upon canonical quantization, the non-vanishing anti-commutation relations between the fermionic operators are given by $\{\bar{\psi}_A^\alpha, \psi_B^\beta\} = \delta^{\alpha\beta} \delta_{AB}$. The $SU(2)$ generators working on these operators are given by $\hat{\mathbf{J}}^a = \bar{\psi}_A^\alpha \sigma_{\alpha\beta}^a \psi_A^\beta / 2$. The $U(M)$ generators are given by:

$$\hat{\mathcal{J}}^n = \bar{\psi}_A^\alpha T_{AB}^n \psi_B^\alpha + c \hat{\mathbb{I}} \delta_0^n , \quad n = 0, 1, \dots, M^2 - 1 . \quad (9.3)$$

The T_{AB}^n with $n > 0$ are the traceless generators of $SU(M)$ subgroup of $U(M)$, and $T_{AB}^0 = \delta_{AB}$ generates the $U(1)$ subgroup of $U(M)$. c is a normal ordering constant that appears as a possible central extension of the $U(1)$. As expected, $[\hat{\mathcal{J}}^n, \hat{\mathbf{J}}^a] = 0$. We take $g > 0$ in what follows and measure quantities in units of g so that $g = 1$.

9.2.1 Spectrum

The Hamiltonian of the system is proportional to the normal ordered square of the angular momentum operator:

$$\hat{H} = - : \bar{\psi}_A^\alpha \sigma_{\alpha\beta}^a \psi_A^\beta \bar{\psi}_B^\gamma \sigma_{\gamma\delta}^a \psi_B^\delta : = -4 : \hat{\mathbf{J}} \cdot \hat{\mathbf{J}} : = -4 \hat{\mathbf{J}} \cdot \hat{\mathbf{J}} + 3\hat{n} , \quad (9.4)$$

where $\hat{n} \equiv \bar{\psi}_A^\alpha \psi_A^\alpha$, commutes with the $\hat{\mathbf{J}}^a$. If we view the index A as a lattice site, the system above is describing two-body $SU(2)$ spin-spin interactions of spin-1/2 fermions between all M possible lattice sites, each with equal strength. From (9.4), it follows that the eigenstates $|J, m; n\rangle$ can be labeled by their total angular momentum J , their angular momentum m in the z -direction and their eigenvalue n with respect to the \hat{n} operator. The energy of $|J, m; n\rangle$ is simply $E = -4J(J+1) + 3n$. For $M > 1$, the ground states $|g\rangle$ are the $(M+1)$ states in the maximally spinning spin- $M/2$ multiplet, whereas the $J = 0$ state with $n = 2M$ has maximal energy. We can construct the full Hilbert space by acting with the $\bar{\psi}_A^\alpha$ operators on the particular $J = 0$ state $|0\rangle$, defined to be the state annihilated by all the ψ_A^α . For instance the ground state with maximal spin- z angular momentum is $|M/2, M/2; M\rangle = \prod_A \bar{\psi}_A^1 |0\rangle$ and has energy $E_g = -M(M-1)$.

For each A we have two states with vanishing angular momentum in the z -direction, and a spin-1/2 doublet. The full Hilbert space can thus be written succinctly as $\mathcal{H} = (0 \oplus 1/2 \oplus 0)^{\otimes M}$. The degeneracies for a given angular momentum in the z -direction can be obtained from the partition function:

$$Z[q] = \text{tr } q^{\sum_A \mathbf{J}_A^z} = \sum_{k=0}^{2M} \binom{2M}{k} q^{M/2 - k/2} \quad (9.5)$$

From the above partition function, we can also obtain the degeneracies of the multiplets with total spin J :

$$d_J = \binom{2M}{M+2J} - \binom{2M}{M+2(J+1)}. \quad (9.6)$$

Indeed, there is exactly one state with $m = M/2$, which is part of the maximally spinning (ground state) multiplet. There are $2M$ states with $m = (M-1)/2$, each of which is part of a spin- $(M-1)/2$ multiplet. However, out of the $M(2M-1)$ states with $m = M/2 - 1$, one is already part of the maximally spinning multiplet, leaving $(2M^2 - M - 1)$ spin- $(M-2)/2$ multiplets. Generalizing this argument to all eigenvalues of $\hat{\mathbf{J}}^z$ yields the formula above. As expected, $\sum_J (2J+1)d_J = 2^{2M}$ and $d_{M/2} = 1$. At large M , using the Stirling approximation, we find a large degeneracy of $2^{2M}/M$ $J = 0$ states. Moreover, for small J/M , we can use the approximations:

$$\begin{aligned} \binom{2M}{M+2J} &\approx \binom{2M}{M} e^{-4J^2/M}, \\ \binom{2M}{M+2(J+1)} &\approx \binom{2M}{M} e^{-4(J+1)^2/M}. \end{aligned} \quad (9.7)$$

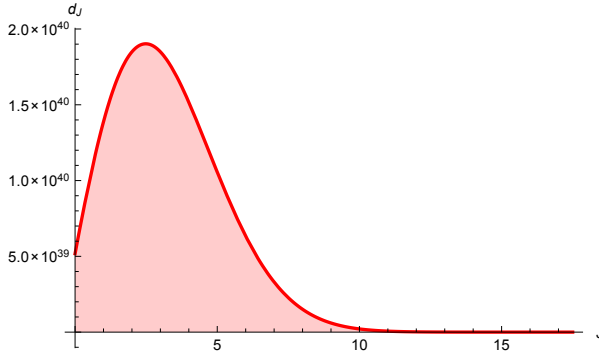


Figure 9.1: Plot of d_J vs. J for $M = 70$.

From these we can derive that d_J peaks at $J \approx \sqrt{M/8}$. We show a plot of the degeneracies d_J in Figure 9.1.

The d_J are the exact degeneracies for the operator $\hat{\tilde{H}} = (\hat{H} - 3\hat{n})$, with eigenvalues $\tilde{E}_J = -4J(J+1)$. At large M , the d_J are also approximately the degeneracies of \hat{H} for several of its lowest lying states. For example, the energy difference between the ground state with $J = M/2$ and the nearest energy level with $J = (M-1)/2$ is $2M$ to leading order. The \hat{n} operator does not split the energies of the $(M+1)$ -fold degenerate states in the ground state multiplet, but it does split the energies of the $2M$ distinct $J = (M-1)/2$ multiplets into two bands of M multiplets separated by an $\mathcal{O}(1)$ amount in energy. Since the energies of both the $J = M/2$ and $J = (M-1)/2$ multiplets are $-M^2$ at large M , to leading order in M the d_J are a good approximation of the degeneracies of \hat{H} for the two lowest lying states. More generally, considerations similar to those leading to (9.6) lead to the formula for the degeneracies of distinct J -multiplets with a given n :

$$d_{J,n} = \binom{M}{\frac{n}{2} + J} \binom{M}{\frac{n}{2} - J} - \binom{M}{\frac{n}{2} + J + 1} \binom{M}{\frac{n}{2} - J - 1}, \quad (9.8)$$

where $n = 2J, 2J+2, \dots, 2M-2J$.² When $J \sim 3M/8$ and below, the energy split among multiplets with the same value of J is large enough to cause overlaps between their energy levels and those of multiplets with different J . For example, the $J = 0$ states have energies ranging between $E_0 \in [0, 6M]$ which can easily be seen to overlap with the energy levels of the $J = 1/2$ states.

² As a simple check, $\sum_n d_{J,n} = d_J$ reproduces (9.6). Furthermore, $\sum_J d_{J,n}(2J+1) = \binom{2M}{n}$, where $J = n/2, n/2-1, \dots$ covers positive integer or half-integer values, depending on whether n is even or odd.

In case we had considered gauging the $U(M)$ symmetry, the spectrum would have changed significantly. For instance, by selecting the normal ordering constant $c = -M$, the only gauge invariant states are the $(M + 1)$ maximally spinning ground states.

9.2.2 Effective theory

We would now like to recast the Euclidean path integral of the theory as a Euclidean path integral of a bosonic (mesonic) variable and understand several features of the model in terms of the bosonic degree of freedom. The Euclidean path integral computes features in the low energy sector the system. For instance, the generating function of vacuum correlation functions is given by:

$$Z[\xi_A^\alpha, \bar{\xi}_A^\alpha] = \int \mathcal{D}\bar{\psi}_A^\alpha \mathcal{D}\psi_A^\alpha e^{-S_E[\bar{\psi}, \psi] - \int d\tau \bar{\xi}_A^\alpha \psi_A^\alpha - \int d\tau \bar{\psi}_A^\alpha \xi_A^\alpha}, \quad (9.9)$$

where the Euclidean action S_E is obtained from $-iS$ by a Wick rotation $t = -i\tau$. Upon introducing an auxiliary three-vector \mathbf{x} and integrating out the Grassmann variables, this can be recast as:

$$Z[\xi_A^\alpha, \bar{\xi}_A^\alpha] = \int \mathcal{D}\mathbf{x} \det(-\partial_\tau + \sigma \cdot \mathbf{x})^M e^{-\int d\tau r^2/4} e^{-\int d\tau \xi_A^\alpha (-\partial_\tau + \sigma \cdot \mathbf{x})_{\alpha\beta}^{-1} \bar{\xi}_A^\beta}, \quad (9.10)$$

where $r = |\mathbf{x}|$. From the partition function we can read off the effective action for the \mathbf{x} degree of freedom:

$$S_{eff} = -M \text{Tr} \log(-\partial_\tau + \sigma \cdot \mathbf{x}) + \int d\tau \frac{r^2}{4}. \quad (9.11)$$

As it stands, the above action is highly non-local in τ . We would like to understand under what conditions this action can be approximated by a small velocity expansion. Generally speaking there is no a priori reason for this to be the case in a quantum system, given that the spectrum is discrete and one cannot continuously change the kinetic energy. However, one may hope that it would be a valid approximation at large M . We will see that this is the case.

Small velocity expansion

It is useful to diagonalize the 2×2 Hermitian matrix $\mathbf{x} \cdot \sigma$ for each τ . Since the σ are traceless, we take some $U \in SU(2)$ such that $U^\dagger \sigma \cdot \mathbf{x} U = r \sigma^z$ for each τ . The U matrix is parameterized by a unit vector $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. Explicitly:

$$U = \begin{pmatrix} \cos \frac{\theta}{2} & e^{-i\phi} \sin \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \end{pmatrix}. \quad (9.12)$$

It then follows that:

$$\det(-\partial_\tau + \sigma \cdot \mathbf{x})^M = e^{M \text{Tr} \log(-\partial_\tau - U^\dagger \dot{U} + r \sigma^z)} . \quad (9.13)$$

Notice that we can transform the above functional determinant under the time reparameterization symmetry

$$\begin{aligned} \tau &\rightarrow f(\tau) , & r(\tau) &\rightarrow \dot{f}(\tau) r(f(\tau)) , & U(\tau) &\rightarrow U(f(\tau)) , \\ e^{M \text{Tr} \log(-\partial_\tau - U^\dagger \dot{U} + r \sigma^z)} &\rightarrow e^{M \text{Tr} \log \dot{f}} e^{M \text{Tr} \log(-\partial_\tau - U^\dagger \dot{U} + r \sigma^z)} . \end{aligned} \quad (9.14)$$

The first factor on the right-hand side of (9.14) is independent of U and r and can be absorbed into the overall normalization of the path integral. The above symmetry can therefore be used to set r to a constant in performing a small velocity expansion of the functional determinant.³ It follows from this that no time derivatives will be generated for r .

We expand (9.13) in powers of $v^a \sigma^a = i U^\dagger \dot{U}$ by expanding the logarithm. The zeroth order term is the effective potential governing r . Going to Fourier space, the computation becomes:

$$V_{eff} = -M \int \frac{d\omega}{2\pi} \log(\omega^2 + r^2) + \frac{r^2}{4} = -M r + \frac{r^2}{4} , \quad (9.15)$$

where we have regulated the ω -integral by differentiating once with respect to r and re-integrating it back while setting the constant of integration to zero. Note that the effective potential is minimized at $r = 2M$ for which $V_{eff}^{(min)} = -M^2$. To leading order in M this agrees with the exact ground state energy of the system $E_g = -M(M+2)$.

The first order term in the velocity expansion is given by:

$$\begin{aligned} S_{kin}^{(1)} &= -M \int \frac{d\omega}{2\pi} (-i\omega + r\sigma^z)_{\alpha\beta}^{-1} i\sigma_{\alpha\beta}^a \tilde{v}^a(0) \\ &= i \frac{M}{2} \int d\tau (1 - \cos \theta) \dot{\phi} , \end{aligned} \quad (9.16)$$

where $\tilde{v}^a(l)$ is the Fourier transform of v^a at frequency l . The linear velocity piece $S_{kin}^{(1)}$ is the phase picked up by a unit charge moving on the surface of a two-sphere, in the presence of a magnetic monopole of strength $M/2$ at the origin.

³ In other words, if we view the symmetries (9.14) as $(0+1)$ -dimensional diffeomorphisms of the worldline, $r(\tau)$ becomes the einbein which can always be gauge fixed to a constant.

Similarly, the quadratic kinetic term is found to be:

$$S_{kin}^{(2)} = M \int d\tau \frac{1}{2r} ((v^x)^2 + (v^y)^2) = M \int d\tau \frac{1}{8r} (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) , \quad (9.17)$$

where in the right-hand side we have expressed the answer in terms of \mathbf{x} , but now written in spherical coordinates. The higher order terms can be similarly computed and they contain even powers of time derivatives of the angular variables divided by one less power of r .⁴

Denoting the characteristic frequency for some particular motion of θ and ϕ by ω_c , the condition that there is a small derivative expansion is:

$$\omega_c \ll r . \quad (9.18)$$

For r near the minimum of the effective potential, we have $\omega_c \ll M$. Hence, for large M there is a parametrically large range of frequencies allowing for a small velocity expansion.

9.2.3 Finite temperature

As was previously noted, the original Grassmann system contains a large number of high energy, i.e. $J = 0$, states at large M . On the other hand the ground state energy is $E_g = -M(M - 1)$. Thus the thermal partition function $Z[\beta] = \text{Tr } e^{-\beta \hat{H}}$ at large β is dominated by the ground states and goes as:

$$\lim_{\beta \rightarrow \infty} Z[\beta] = (M + 1) e^{M(M-1)\beta} , \quad (9.19)$$

whereas at small β we have simply the dimension of the Hilbert space:

$$\lim_{\beta \rightarrow 0} Z[\beta] = 2^{2M} . \quad (9.20)$$

The transition between these two behaviors occurs at $\beta \sim 1/M$.

We now consider the finite temperature partition function as a Euclidean path integral over \mathbf{x} . We must integrate out the Grassmann numbers with anti-periodic boundary conditions along the thermal circle. In analogy to previous calculations, we can compute the thermal effective potential. What changes is that the ω -integrals are replaced by sums over the thermal frequencies $\omega_n = 2\pi(n + 1/2)/\beta$ with $n \in \mathbb{Z}$. The thermal effective potential thus becomes:

$$V_{eff}(\beta) = -\frac{M}{\beta} \sum_{n \in \mathbb{Z}} \log(\omega_n^2 + r^2) + \frac{r^2}{4} = -\frac{2M}{\beta} \log \cosh \frac{r\beta}{2} + \frac{r^2}{4} . \quad (9.21)$$

⁴ In appendix B of [3] we consider a modified vector model where the leading kinetic piece is (9.17).

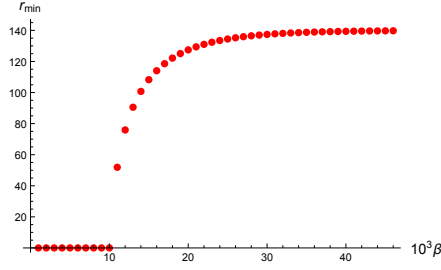


Figure 9.2: Plot of value of r minimizing $V_{eff}(\beta)$ vs. $10^3 \times \beta$ for $M = 70$. Notice that the value stays close to $2M = 140$ all the way down to $\beta \sim 1/M$.

As before, the sum has been regulated by differentiating with respect to r .

For large β , the minimum of V_{eff} is at $r = 2M$ as for the zero temperature analysis. We can find the critical point for r in a large β expansion. To first order:

$$r = 2M (1 - 2e^{-2M\beta} + \dots) . \quad (9.22)$$

From this we see the tendency of r to decrease upon increasing the temperature. At small β , we can Taylor expand:

$$V_{eff}(\beta) = \frac{r^2}{4} - \frac{\beta}{4} M r^2 + \mathcal{O}(\beta^2) . \quad (9.23)$$

We see that for $\beta \lesssim 1/M$ the thermal potential is minimized at $r = 0$. In Figure 9.2 we show a plot for the values of r minimizing $V_{eff}(\beta)$ as we vary β .

When r is near zero, we can no longer assume that the kinetic contributions are small and thus our analysis breaks down. This as an indication that the high temperature phase does not have a reliable small velocity description in terms of \mathbf{x} . Instead, the correct description requires taking into account the full set of Grassmann degrees of freedom.

9.2.4 Bloch coherent state path integral

So far we have introduced the variable \mathbf{x} as a convenient integration variable to capture correlations in the vacuum state and thermal properties. Here we would like to point out that in a fixed large angular momentum sector, there is some more significance to \mathbf{x} .

Following Bloch, we define a collection of coherent states built from the state $|v\rangle$, which has the lowest angular momentum in the z -direction and hence

is also a minimal energy state. In other words $|v\rangle = \prod_A \bar{\psi}_A^2 |0\rangle$. We can act on $|v\rangle$ with the spin raising operator $\hat{\mathbf{J}}^+ = \hat{\mathbf{J}}^x + i\hat{\mathbf{J}}^y$ to generate states in the maximally spinning multiplet,

$$|\bar{z}\rangle = \frac{1}{(1+z\bar{z})^{M/2}} e^{\bar{z}\hat{\mathbf{J}}^+} |v\rangle, \quad z \in \mathbb{C}. \quad (9.24)$$

These states are not orthogonal, but they constitute an over-complete basis of the Hilbert space of the maximally spinning multiplet,

$$\langle w|\bar{z}\rangle = \frac{(1+w\bar{z})^M}{(1+w\bar{w})^{M/2}(1+z\bar{z})^{M/2}}, \quad \int d^2z \frac{M+1}{\pi(1+z\bar{z})^2} |\bar{z}\rangle \langle z| = \mathbb{I}. \quad (9.25)$$

The purpose of these states is to describe, with minimal uncertainty, points on the S^2 of spin directions. Indeed, the angular momentum expectation value defines a point on S^2 – through the stereographic projection – with decreasing uncertainty in the large M limit

$$\mathbf{J}^a \equiv \langle z|\hat{\mathbf{J}}^a|\bar{z}\rangle = \frac{M}{2(1+|z|^2)} (z + \bar{z}, i(\bar{z} - z), |z|^2 - 1), \quad (9.26)$$

$$\frac{\langle z|(\hat{\mathbf{J}}^a - \mathbf{J}^a)^2|\bar{z}\rangle}{\langle z|\hat{\mathbf{J}}^a|\bar{z}\rangle^2} = \frac{2}{M}.$$

One may ask about transition amplitude between two such states: $\langle z_N|e^{-iT\hat{H}}|\bar{z}_0\rangle$ for some given Hamiltonian \hat{H} built out of the $\hat{\mathbf{J}}^a$. The result is [227, 228]:

$$\langle z_N|e^{-iT\hat{H}}|\bar{z}_0\rangle = \int \mathcal{D}z\mathcal{D}\bar{z} \frac{(M+1)}{\pi(1+z\bar{z})^2} e^{iS(z,\bar{z})}, \quad (9.27)$$

with

$$S = i\frac{M}{2} \int dt \frac{(z\dot{\bar{z}} - \dot{z}\bar{z})}{1+z\bar{z}} - \int dt H(z,\bar{z}), \quad (9.28)$$

where $H(z,\bar{z}) \equiv \langle z|\hat{H}|\bar{z}\rangle$. The boundary conditions are $z(T) = z_N$ and $\bar{z}(0) = \bar{z}_0$. For our particular choice of Hamiltonian, $H(z,\bar{z}) = -M(M-1)$. Given the first order form of the action (9.28) appearing in the path integral (9.27), the complex variable z can be viewed as a complex coordinate parameterizing a two-dimensional phase space. From the linear velocity piece in (9.28) we note that the phase space is curved and compact, with Kähler metric:

$$ds^2 = 2M \frac{dzd\bar{z}}{(1+z\bar{z})^2}. \quad (9.29)$$

This is the Fubini-Study metric on $\mathbb{CP}^1 \cong S^2$, and we occasionally refer to it as the Bloch sphere. The symplectic form is given by the Kähler form

and the large M limit plays the role of the small Planck constant limit. Time evolution of a function $A(z, \bar{z})$ in the emergent classical phase space is governed by the Poisson bracket, i.e. $\dot{A}(z, \bar{z}) = \{A(z, \bar{z}), H(z, \bar{z})\}_{p.b.} = i M^{-1} (1 + z\bar{z})^2 (\partial_{\bar{z}} H \partial_z A - \partial_z H \partial_{\bar{z}} A)$. The $SU(2)$ symmetry of the original Grassmann model acts on z as:

$$z \rightarrow (\alpha z + \beta)(\gamma z + \delta)^{-1}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^\dagger = \mathbb{I}_{2 \times 2}. \quad (9.30)$$

Since the classical phase space has finite volume, we recover the fact that the underlying system has a finite number of ground states. The complex coordinate (z, \bar{z}) can be related to the spherical coordinates introduced in (9.12) by identifying the expectation value (9.26) with the bosonic variable \mathbf{x} introduced in the previous section. The stereographic projection then gives $z = e^{i\phi} \cot \theta/2$. With this identification, the linear velocity term in (9.28) becomes precisely the one found in (9.16). Thus, we see that certain transition amplitudes are captured by a real time path integral between different points localized on an S^2 . This allows for physical interpretation of the (θ, ϕ) coordinates as real time degrees of freedom, rather than merely integration variables.

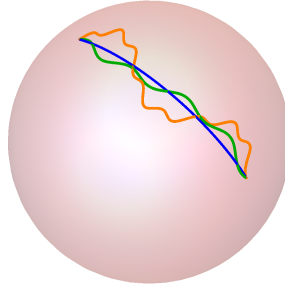


Figure 9.3: Schematic plot of classical and nearby trajectories on the Bloch sphere for some $H(z, \bar{z})$, contributing to the path integral (9.28). At large M the classical trajectory dominates.

We can quantize this low energy effective theory to leading order in the velocity expansion. This becomes the quantum mechanics of an electrically charged particle with unit charge. Its motion is confined to a unit sphere in the presence of a magnetic monopole of strength $M/2$ at the origin. Thus, to leading order in M the ground states are given by the M lowest Landau levels, each with energy $E_g = -M^2$ for our choice of Hamiltonian. Due to the Dirac quantization condition, we recover that M must be an integer.

We have seen how certain low energy features in the original Grassmann theory are described in the language of the effective bosonic degree of freedom \mathbf{x} . Instead of maximally spinning states built out of anti-commuting creation operators, we have lowest Landau levels of a charged particle. The energies (at least in the low energy regime) are registered by the absolute value of \mathbf{x} . We have observed the breakdown of the bosonic effective theory at high temperatures. Certain features were particular to our model. But others such as the presence of linear velocity terms and the absence of a kinetic term for r may be general features of a larger class of models. At this point we proceed to generalize these observations to the case where we have a matrix worth of Grassmann degrees of freedom.

9.3 Matrix model

The goal of this section is to analyze a matrix version of the vector model studied above. Given that the model is more complicated, we will not be able to attain as explicit a description, however we will uncover and generalize several of the features found in the vector model.

9.3.1 Action and Hamiltonian

Our degrees of freedom are now $2MN$ complex rectangular Grassmann matrices, $\bar{\psi}_{iA}^\alpha$ and ψ_{Ai}^α , with $A = 1, \dots, M$ and $i = 1, \dots, N$. As before, α is an $SU(2)$ spinor index. The dimension of the Hilbert space now becomes 2^{2NM} . The Grassmann elements obey the anti-commutation relations $\{\psi_{Ai}^\alpha, \bar{\psi}_{jB}^\beta\} = \delta^{\alpha\beta} \delta_{ij} \delta_{AB}$.

We will focus on the following action:⁵

$$S = \int dt i \bar{\psi}_{iA} \partial_t \psi_{Ai} + g (\bar{\psi}_{iA} \sigma^a \psi_{Aj}) (\bar{\psi}_{jB} \sigma^a \psi_{Bi}) . \quad (9.31)$$

When $N = 1$, the above action reduces to the one analyzed in the previous section. The model exhibits a $U(M) \times SU(N) \times SU(2)$ global symmetry. The $SU(2)$ acts by simultaneously rotating all the Grassmann elements. The capitalized index of $(\bar{\psi}_{iA}^\alpha \psi_{Ai}^\alpha)$ transforms in the (anti-)fundamental representation of $U(M)$ whereas the lower case index transforms in the (anti-)fundamental of $SU(N)$.

The Hamiltonian of the model is given by:

$$\hat{H} = -g \sum_{i,j,A,B} : \bar{\psi}_{iA} \sigma \psi_{Aj} \bar{\psi}_{jB} \sigma \psi_{Bi} : \quad (9.32)$$

If we view the A index as a lattice site, our system describes $SU(2)$ spin-spin interactions of the spin-1/2 fermions. But now the fermions are labeled by an additional quantum number, the color index $i = 1, 2, \dots, N$, which can be exchanged through the interaction. Since interactions between all lattice sites have the same strength, the model exhibits no notion of spatial locality.

We will analyze $g > 0$ and from now on choose units setting $g = 1$. Unlike the vector case previously studied, the combinatorial problem of finding the exact spectrum of \hat{H} seems to be rather difficult and we have not solved it. Instead, we will try to extract information about the low energy sector of the theory by going to an effective description in terms of bosonic matrices. Before doing so, we will establish some further properties about the operator algebra.

U(2N) operator algebra

The analogues of the spin operators $\hat{\mathbf{J}}^a$ studied in the previous section are the $U(M)$ invariant $N \times N$ spin matrix operators: $\hat{\mathbf{S}}_{ij}^a = \sum_A (\bar{\psi}_{iA} \sigma^a \psi_{Aj})/2$. These operators transform as vectors in the three-dimensional real representation of $SU(2)$, as well as in the adjoint of the $SU(N)$. Introducing an additional operator $\hat{\mathbf{S}}_{ij}^0 = \sum_A (\bar{\psi}_{iA} \sigma^0 \psi_{Aj})/2$, with σ^0 the 2×2 identity matrix, we have the following closed operator algebra:

$$[\hat{\mathbf{S}}_{ij}^a, \hat{\mathbf{S}}_{kl}^b] = \frac{1}{2} \delta^{ab} (\delta_{kj} \hat{\mathbf{S}}_{il}^0 - \delta_{il} \hat{\mathbf{S}}_{kj}^0) + \frac{i}{2} \epsilon^{abc} (\delta_{kj} \hat{\mathbf{S}}_{il}^c + \delta_{il} \hat{\mathbf{S}}_{kj}^c) ,$$

⁵ We have and will continue to suppress the $SU(2)$ spinor index in ψ_{Ai}^α to avoid cluttering of indices.

$$\begin{aligned}
[\hat{\mathbf{S}}_{ij}^0, \hat{\mathbf{S}}_{kl}^a] &= \frac{1}{2} \left(\delta_{kj} \hat{\mathbf{S}}_{il}^a - \delta_{il} \hat{\mathbf{S}}_{kj}^a \right) , \\
[\hat{\mathbf{S}}_{ij}^0, \hat{\mathbf{S}}_{kl}^0] &= \frac{1}{2} \left(\delta_{kj} \hat{\mathbf{S}}_{il}^0 - \delta_{il} \hat{\mathbf{S}}_{kj}^0 \right) .
\end{aligned} \tag{9.33}$$

The N diagonal components of the $\hat{\mathbf{S}}_{ij}^a$ generate N copies of the usual $su(2)$ algebra. The above operators can be arranged in a $2N \times 2N$ Hermitian matrix $\sigma_{\alpha\beta}^\mu \otimes \hat{\mathbf{S}}_{ij}^\mu$ (with $\mu = \{0, x, y, z\}$ summed over) and hence they generate a $u(2N)$ algebra. They act as $\psi_{Ai}^\alpha \rightarrow \psi_{Ai}^\alpha G_{ij}^{\alpha\beta}$ and $\bar{\psi}_{iA}^\alpha \rightarrow (G_{ij}^{\alpha\beta})^{-1} \bar{\psi}_{jB}^\beta$ with $G_{ij}^{\alpha\beta} = e^{i\lambda_{ij}^{\alpha\beta}} \in U(2N)$ and $\lambda_{ij}^{\alpha\beta} = \lambda_{ij}^\mu \sigma_{\alpha\beta}^\mu$ the elements of a $2N \times 2N$ Hermitian matrix.

The $U(2N)$ symmetry manifestly commutes with the $U(M)$ group and preserves the anti-commutation relations between the ψ_{Ai}^α and $\bar{\psi}_{iA}^\alpha$. Our Hamiltonian (9.32) does not commute with the full $U(2N)$ but rather the $U(N)$ diagonal subgroup generated by the $\hat{\mathbf{S}}_{ij}^0$. When $N = 1$, the $U(2N)$ algebra becomes nothing more than the global $SU(2)$ symmetry of the vector model, which not only commutes with the $U(M)$ global symmetry but also with the Hamiltonian.

9.3.2 Effective theory

We introduce three $N \times N$ Hermitian bosonic matrices $\mathbf{\Sigma}_{ij}^a = (\mathbf{\Sigma}_{ij}^x, \mathbf{\Sigma}_{ij}^y, \mathbf{\Sigma}_{ij}^z)$. In analogy with the vector case, we introduce them as auxiliary variables which are given on-shell by $\mathbf{\Sigma}_{ij}^a = 2\hat{\mathbf{S}}_{ij}^a$. Upon integrating out the ψ_{Ai}^α , the generating function of vacuum correlations of ψ and $\bar{\psi}$ can be expressed as a Euclidean path integral over the $\mathbf{\Sigma}_{ij}$:

$$Z[\xi_{Ai}^\alpha, \bar{\xi}_{iA}^\alpha] = \int \mathcal{D}\mathbf{\Sigma} e^{M \text{Tr} \log(-\partial_\tau + \mathbf{R}) - \frac{1}{4} \text{tr} \int d\tau \mathbf{\Sigma} \cdot \mathbf{\Sigma}} e^{\int d\tau \bar{\xi}_{iA}^\alpha (-\partial_\tau + \mathbf{R})_{ij, \alpha\beta}^{-1} \xi_{Aj}^\beta} . \tag{9.34}$$

We have defined $\mathbf{R} \equiv \mathbf{\Sigma}^x \otimes \sigma^x + \mathbf{\Sigma}^y \otimes \sigma^y + \mathbf{\Sigma}^z \otimes \sigma^z$. We also denote the full functional trace by ‘Tr’ and reserve the ‘tr’ symbol for the ordinary matrix trace. It follows from this definition that $\text{tr} \mathbf{R} = 0$. The global $SU(N)$ symmetry acts as $\mathbf{\Sigma} \rightarrow U \mathbf{\Sigma} U^\dagger$. Also, $\mathbf{\Sigma}$ transforms as in the three-dimensional (vector) representation of the global $SU(2)$ symmetry group. We can also write down the generating function for vacuum correlations of the composite spin-matrix operator $\hat{\mathbf{S}}_{ij}^a$. These are computed by the correlation functions of $\mathbf{\Sigma}_{ij}$ itself:

$$Z[\mathbf{J}_{ij}^a] = \int \mathcal{D}\mathbf{\Sigma} e^{M \text{Tr} \log(-\partial_\tau + \mathbf{R}) - \frac{1}{4} \text{tr} \int d\tau \mathbf{\Sigma} \cdot \mathbf{\Sigma}} e^{\frac{1}{4} \text{tr} \int d\tau \mathbf{J} \cdot \mathbf{\Sigma} - \frac{1}{16} \text{tr} \int d\tau \mathbf{J} \cdot \mathbf{J}} , \tag{9.35}$$

where \mathbf{J}_{ij}^a are sources for the $\hat{\mathbf{S}}_{ij}^a$. It is worth noting that, unlike the $N = 1$ case, the $\hat{\mathbf{S}}_{ij}^a$ no longer commute with the Hamiltonian and thus non-trivial time correlations amongst them may exist.

We now proceed to study the validity and properties of the ‘small velocity’ expansion of $\det(-\partial_\tau + \mathbf{R}) = \exp[\text{Tr} \log(-\partial_\tau + \mathbf{R})]$. Since \mathbf{R} is a $2N \times 2N$ Hermitian matrix, we can diagonalize it as $U^\dagger \mathbf{R} U = \lambda$ with $\lambda = \text{diag}[\lambda_1, \dots, \lambda_{2N}]$, $U \in U(2N)$ and $\lambda_n \in \mathbb{R}$. Note that due to the tracelessness of \mathbf{R} , not all λ_n can have the same sign. Similar to the $N = 1$ case, in the diagonal \mathbf{R} frame, we can write the functional determinant as:

$$\text{Tr} \log(-\partial_\tau + \mathbf{R}) = \text{Tr} \log(-\partial_\tau - U^\dagger \dot{U} + \lambda) . \quad (9.36)$$

With the above expression we can again use the time reparameterization symmetry

$$\tau \rightarrow f(\tau) , \quad \lambda_n(\tau) \rightarrow f'(\tau) \lambda_n(f(\tau)) , \quad U(\tau) \rightarrow U(f(\tau)) , \quad (9.37)$$

to see that the effective action will be independent of $\dot{\lambda}_n$, analogous to how the vector model is independent of \dot{r} . Using the propagator:

$$G(\omega) = \text{diag} \left[(-i\omega + \lambda_1)^{-1}, \dots, (-i\omega + \lambda_{2N})^{-1} \right] , \quad (9.38)$$

we can expand the logarithm in powers of the Hermitian matrix $v = iU^\dagger \dot{U}$. Each term in the expansion will be endowed with a $U(2N)$ symmetry taking $U^\dagger \dot{U} \rightarrow \Lambda^\dagger (U^\dagger \dot{U}) \Lambda$ and $\lambda \rightarrow \Lambda^\dagger \lambda \Lambda$ with $\Lambda \in U(2N)$.

The linear velocity contribution to the effective action is:

$$\begin{aligned} S_{kin}^{(1)} &= -iM \text{tr} \int \frac{d\omega}{2\pi} G(\omega) \tilde{v}(0) \\ &= -i \frac{M}{2} \sum_m \text{sgn}(\lambda_m) \int d\tau [iU^\dagger \dot{U}]_{mm} . \end{aligned} \quad (9.39)$$

The $\tilde{v}(l)$ is the Fourier transform of v at frequency l . To define the above ω -integral we have put a cutoff at large ω , performed the exact integration and then taken the large cutoff limit. The kinetic piece containing two time derivatives in $U(\tau)$ is given by:

$$\begin{aligned} S_{kin}^{(2)} &= -\frac{M}{2} \text{tr} \int \frac{d\omega dl}{(2\pi)^2} G(\omega) \tilde{v}(l) G(\omega) \tilde{v}(-l) \\ &= \frac{M}{2} \sum_{n,m} \int d\tau [iU^\dagger \dot{U}]_{nm} \Lambda_{mn} [iU^\dagger \dot{U}]_{mn} , \end{aligned} \quad (9.40)$$

with $\Lambda_{mn} = 1/|\lambda_m - \lambda_n|$ and the sum running only over the pairs (n, m) for which λ_n and λ_m have opposite signs. The reason why only pairs of λ_m with opposite sign appear in the sum is that the integral appearing in (9.40):

$$\mathcal{I}_{mn} = \int \frac{d\omega}{2\pi} \frac{1}{(-i\omega + \lambda_m)} \frac{1}{(-i\omega + \lambda_n)} \quad (9.41)$$

vanishes whenever λ_n and λ_m have the same sign. It is interesting to note that the effective kinetic piece of the theory, and hence what we mean by the dynamical content, depends on the particular distribution of eigenvalues λ_n .

Having obtained expressions for the first few velocity dependent terms in the effective action, we can estimate when the low velocity expansion is valid. Denoting the characteristic frequency for some motion as ω_c , then in order for $S_{kin}^{(1)}$ to be large compared to $S_{kin}^{(2)}$ one requires:

$$\omega_c \ll \frac{\lambda_n}{N}. \quad (9.42)$$

The factor of N stems from the fact that $S_{kin}^{(2)}$ has an additional matrix index to be summed over that was not present in the vector model previously studied. In what follows we will see that the effective potential is minimized for $\lambda_m \sim M$. Thus, in the limit $M \gg N$, we can have a large range of allowed ω_c (in units where $g = 1$). If instead M does not scale with N and we take the large N limit, the window of allowed ω_c shrinks to zero.

Since the global symmetry group of the theory, for our choice of Hamiltonian, is *not* the full $U(2N)$, the situation is not as simple as the $N = 1$ case. For instance, the Σ measure in the path integral is not $U(2N)$ invariant. Moreover, it is in general complicated to quantify how the Σ matrices are encoded in the λ_n eigenvalues and U matrices. In what follows we express several parts of the effective action directly in terms of the Σ .

Effective potential

We would now like to focus on the effective potential V_{eff} for Σ . In order to compute this we can take Σ to be time independent. V_{eff} must respect the $SU(N) \times SU(2)$ symmetries. For instance it can contain a piece which is the trace of a function of the $SU(2)$ invariant matrix $\Sigma \cdot \Sigma$. Moreover, when the Σ are diagonal (or when they all commute with each other), it must reproduce N copies of the potential (9.15) we found in the vector model. Finally, the piece of V_{eff} originating from the functional determinant must scale linearly in Σ .

We can write a general expression by noting that:

$$\det_{2N \times 2N} (-i\omega + \mathbf{R}) = \prod_{n=1}^{2N} (-i\omega + \lambda_n) , \quad (9.43)$$

is the characteristic polynomial for matrix \mathbf{R} with eigenvalues λ_n . We must also take the product over all ω , a procedure which must be regulated. For each λ_n , we can express the product over the ω as the exponential of an integral over the logarithm:

$$\frac{1}{2} \int \frac{d\omega}{2\pi} \log(\omega^2 + \lambda_n^2) = \frac{|\lambda_n|}{2} . \quad (9.44)$$

To define the above integral,⁶ we have subtracted the integral of $\log(\omega^2)$. Putting things together:

$$V_{eff} = -\frac{M}{2} \sum_{n=1}^{2N} |\lambda_n| + \frac{1}{4} \text{tr} \mathbf{\Sigma} \cdot \mathbf{\Sigma} = -\frac{M}{2} \text{tr} \sqrt{\mathbf{R}^2} + \frac{1}{4} \text{tr} \mathbf{\Sigma} \cdot \mathbf{\Sigma} . \quad (9.45)$$

As expected, V_{eff} is invariant under both the $SU(N)$ and $SU(2)$ global symmetries. It is instructive to write the $2N \times 2N$ matrix \mathbf{R}^2 explicitly:

$$\mathbf{R}^2 = \begin{pmatrix} \mathbf{\Sigma} \cdot \mathbf{\Sigma} - i[\mathbf{\Sigma}^x, \mathbf{\Sigma}^y] & [\mathbf{\Sigma}^z, \mathbf{\Sigma}^x + i\mathbf{\Sigma}^y] \\ -[\mathbf{\Sigma}^z, \mathbf{\Sigma}^x - i\mathbf{\Sigma}^y] & \mathbf{\Sigma} \cdot \mathbf{\Sigma} + i[\mathbf{\Sigma}^x, \mathbf{\Sigma}^y] \end{pmatrix} . \quad (9.46)$$

From the above expression, it immediately follows that $\text{tr} \mathbf{R}^2 = 2 \text{tr} \mathbf{\Sigma} \cdot \mathbf{\Sigma}$. However, this does *not* imply that $\text{tr} \sqrt{\mathbf{R}^2} = 2 \text{tr} \sqrt{\mathbf{\Sigma} \cdot \mathbf{\Sigma}}$ unless all the $\mathbf{\Sigma}$ commute amongst each other. Thus, we see how the commutator interaction enters the potential. If it happens that the $\mathbf{\Sigma}$ are almost commuting, we can perform a matrix Taylor expansion of $\text{tr} \sqrt{\mathbf{R}^2}$, which to leading order gives:

$$-\frac{M}{2} \text{tr} \sqrt{\mathbf{R}^2} \approx -M \text{tr} \sqrt{\mathbf{\Sigma} \cdot \mathbf{\Sigma}} + \frac{M}{16} \text{tr} (\mathbf{\Sigma} \cdot \mathbf{\Sigma})^{-1/2} i[\mathbf{\Sigma}^a, \mathbf{\Sigma}^b] (\mathbf{\Sigma} \cdot \mathbf{\Sigma})^{-1} i[\mathbf{\Sigma}^a, \mathbf{\Sigma}^b] + \dots \quad (9.47)$$

The indices (a, b) run over all distinct pairs of (x, y, z) , thus rendering the expression $SO(3)$ invariant. Since the Hermitian matrix $\mathbf{\Sigma} \cdot \mathbf{\Sigma}$ has positive eigenvalues, and the commutator $i[\mathbf{\Sigma}^a, \mathbf{\Sigma}^b]$ is Hermitean, we see that non-zero commutations cost potential energy. Thus, at least locally the potential (9.45) is minimized when the $\mathbf{\Sigma}$ mutually commute (which means, in turn, that we can mutually diagonalize the $\mathbf{\Sigma}$). In this approximation, we can estimate the

⁶ One may be concerned about the discontinuity of the first derivative at $\lambda_n = 0$. However, the expression agrees with what we expect of the determinant $\prod_{\omega} (1 + \lambda_n^2/\omega^2)$. Namely, it should equal one when $\lambda_n = 0$, it should be symmetric under $\lambda_n \rightarrow -\lambda_n$ and have an exponent linear in λ_n . Moreover, one can check that at any non-zero temperature T for which $\omega \rightarrow 2\pi T(n + 1/2)$ with $n \in \mathbb{Z}$, the kink at $\lambda_n = 0$ smoothenes out.

minimum value of V_{eff} as the first term in the expansion (9.47). The problem we want to solve becomes a saddle point approximation of the following matrix integral for $M \gg N$:

$$Z[\Sigma] = \int d\Sigma^x d\Sigma^y d\Sigma^z e^{M \text{tr} \sqrt{\Sigma \cdot \Sigma} - \text{tr} \Sigma \cdot \Sigma / 4} . \quad (9.48)$$

In order to obtain the saddle point equation for the eigenvalues, we first introduce a delta function $\delta(\rho - \Sigma \cdot \Sigma)$ and integrate out the Σ , such that we remain with an integral over the $N \times N$ Hermitian ρ matrix. Upon diagonalizing ρ , and including the Vandermonde contribution, we can obtain the potential for its eigenvalues $\rho_i \geq 0$. It is convenient at this point to rescale $\rho_i = M^2 \tilde{\rho}_i$. We find:

$$V_{eff}[\tilde{\rho}_i] = - \sum_{j \neq i} \log |\tilde{\rho}_i - \tilde{\rho}_j| - M^2 \sum_i \left(\sqrt{\tilde{\rho}_i} - \frac{\tilde{\rho}_i}{4} + \frac{2N}{M^2} \log \tilde{\rho}_i \right) , \quad (9.49)$$

up to an additive constant of order $N^2 \log M$. The $\log \tilde{\rho}_i$ contribution comes from the measure of the path integral: there is a Jacobian when changing variables from the Σ matrices to the ρ matrix. The saddle point equation governing the eigenvalues is:

$$\sum_{j \neq i}^N \frac{1}{\tilde{\rho}_i - \tilde{\rho}_j} = -\frac{2N}{\tilde{\rho}_i} - M^2 \left(\frac{1}{2\sqrt{\tilde{\rho}_i}} - \frac{1}{4} \right) . \quad (9.50)$$

To leading order in a large M expansion (taking M to be much larger than N) we can consider $\tilde{\rho}_i$ to be peaked around $\tilde{\rho}_i \sim 4$. Expanding about $\tilde{\rho}_i = 4 + \delta_i$ for small δ_i , and keeping the leading term only, we have:

$$\sum_{j \neq i}^N \frac{1}{\delta_i - \delta_j} = \frac{M^2}{32} \delta_i . \quad (9.51)$$

For large⁷ N , the above eigenvalue equation is solved by the Wigner semicircle distribution [210] and has compact support in the interval $(\sqrt{32N}/M) \times [-1, 1]$. Thus, going back to the original eigenvalues, we see that they are peaked around $\rho_i \approx 4M^2$ with a width of order $\sqrt{N}M$. We can approximate the ground state energy to be $V_{eff}^{(min)} \approx -M^2N$. It would be interesting to study subleading corrections, due to the repulsion of eigenvalues from the Vandermonde, but we will not do so here.

There is a slightly more efficient way to see the above. Using the property $\text{tr} \mathbf{R}^2 = 2 \text{tr} \Sigma \cdot \Sigma$ we can write the effective potential (9.45) completely in terms

⁷ We are considering here the situation where both M and N are large but $M \gg N$.

of the eigenvalues of \mathbf{R} as:

$$V_{eff} = \frac{1}{2} \sum_{n=1}^{2N} \left(-M|\lambda_n| + \frac{\lambda_n^2}{4} \right) . \quad (9.52)$$

Again, at least in the limit $M \gg N$ where we can ignore the effects of the matrix measure, we find $V_{eff}^{(min)} \approx -M^2 N$ as before.

Linear velocity term

We consider the linear velocity term for the matrix model. The simplest case occurs when the Σ_{ij} matrix is diagonal, i.e. $\Sigma_{ij} = \mathbf{x}_i \delta_{ij}$ with $i = 1, \dots, N$. In this case, we simply find a sum of N terms (one for each \mathbf{x}_i) each identical with the vector case. Each will have their own $M + 1$ lowest Landau levels. Generally, however, the Σ^a will not be mutually diagonalizable. Inspired by the expression (9.28), we claim that the linear velocity term is given by:

$$S_{kin}^{(1)} = i \frac{M}{2} \text{tr} \int dt \left[\dot{Z}^\dagger (\mathbb{I} + ZZ^\dagger)^{-1} Z - Z^\dagger (\mathbb{I} + ZZ^\dagger)^{-1} \dot{Z} \right] , \quad (9.53)$$

where Z_{ij} is a complex $N \times N$ matrix. The stereographic map (9.26) relating z to a point on the Bloch sphere is generalized to:

$$\begin{aligned} \Sigma^x + i\Sigma^y &\equiv 2M Z (\mathbb{I} + Z^\dagger Z)^{-1} , \\ \Sigma^x - i\Sigma^y &\equiv 2M Z^\dagger (\mathbb{I} + ZZ^\dagger)^{-1} , \\ \Sigma^z &\equiv M \left[\mathbb{I} - (\mathbb{I} + ZZ^\dagger)^{-1} - (\mathbb{I} + Z^\dagger Z)^{-1} \right] .y \end{aligned} \quad (9.54)$$

In order to verify that $\Sigma^a = (\Sigma^a)^\dagger$ it is useful to take advantage of identities such as: $(\mathbb{I} + ZZ^\dagger)^{-1} Z = Z(\mathbb{I} + Z^\dagger Z)^{-1}$. Naturally, when $N = 1$ our expression (9.53) reduces to the expression (9.28). It is also time reparameterization invariant under $\tau \rightarrow f(\tau)$ and $Z_{ij}(\tau) \rightarrow Z_{ij}(f(\tau))$. Moreover, our expression is invariant under the global $SU(N)$, under which $Z \rightarrow \Lambda Z \Lambda^\dagger$, with $\Lambda \in SU(N)$. In fact, as we shall see in the next subsection, (9.53) is invariant under a larger group $U(2N)$ acting as:

$$Z \rightarrow (AZ + B)(CZ + D)^{-1} , \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix}^\dagger = \mathbb{I}_{2N \times 2N} . \quad (9.55)$$

where A, B, C and D are $N \times N$ matrices. The $U(2N)$ invariance is in agreement with our observation that terms stemming from the functional determinant

(9.36) exhibit a $U(2N)$ symmetry. This generalizes the $SU(2)$ symmetry (9.30) that is present in the $N = 1$ case. Recall that in the $N = 1$ case, the linear velocity term only depended on two of the three variables in \mathbf{x} . Analogously, our expression (9.53) only depends on $2N^2$ of the $3N^2$ variables in the three Hermitian matrices Σ^a .

9.3.3 Berezin coherent states

As in the vector case, the matrix action (9.53) can stem from a curved phase space endowed with a Kähler structure. These compact Kähler manifolds were studied extensively by Berezin [204]. The Kähler metric is given by:

$$ds^2 = M \operatorname{tr} dZ (\mathbb{I} + ZZ^\dagger)^{-1} dZ^\dagger (\mathbb{I} + Z^\dagger Z)^{-1} , \quad (9.56)$$

where c is a normalization constant. The Kähler potential is given by:

$$K = M \log (\mathbb{I} + ZZ^\dagger) . \quad (9.57)$$

This potential transforms under the $U(2N)$ isometry (9.55) as

$$K \rightarrow K - M \log \det(Z^\dagger C^\dagger + D^\dagger) - M \log \det(CZ + D) , \quad (9.58)$$

leaving the metric (9.56) invariant. It is the natural generalization of the $N = 1$ case.

More precisely, what Berezin shows [204] is that there exist a collection of coherent states, analogous to the Bloch coherent states, parameterized by a complex matrix Z_{ij} . Explicitly:

$$|Z_{ij}^\dagger\rangle = \frac{e^{Z_{ij}^\dagger \hat{\mathbf{S}}_{ji}^+}}{\det(\mathbb{I} + Z^\dagger Z)^{M/2}} |v\rangle , \quad \hat{\mathbf{S}}_{ij}^\pm = \hat{\mathbf{S}}_{ij}^x \pm i\hat{\mathbf{S}}_{ij}^y , \quad (9.59)$$

where the state $|v\rangle$ is the state annihilated by all ψ_{Ai}^1 and $\bar{\psi}_{iA}^2$ operators. It can be expressed as $|v\rangle = \prod_{A,i} \bar{\psi}_{iA}^2 |0\rangle$, where $|0\rangle$ is the state that is annihilated by all the ψ_{Ai}^α operators. Consequently $|v\rangle$ is annihilated by $\hat{\mathbf{S}}_{ij}^-$. The overlap between two Berezin coherent states is given by:

$$\langle W_{ij} | Z_{ij}^\dagger \rangle = \frac{\det (\mathbb{I} + WZ^\dagger)^M}{\det (\mathbb{I} + W^\dagger W)^{M/2} \det (\mathbb{I} + Z^\dagger Z)^{M/2}} . \quad (9.60)$$

At large M the quantum evolution of a certain class of $U(M)$ invariant operators in the Grassmann theory becomes approximately classical with an emergent curved phase space [204] , the geometry of which is described by the Kähler

metric (9.56). The role of large M becomes that of the small Planck constant. The classical Hamiltonian, governing the time evolution of functions on the emergent phase space, is given by $H[Z, Z^\dagger] = \langle Z | \hat{H} | Z^\dagger \rangle$. The volume of the emergent classical phase space computes the number of quantum states obtained upon quantizing it. The number of quantum states was computed in [229]. The result reads:

$$\dim \mathcal{H}_K = \prod_{j=1}^N \frac{\Gamma[N + M + j] \Gamma[j]}{\Gamma[N + j] \Gamma[M + j]} . \quad (9.61)$$

We can study the behavior of $\dim \mathcal{H}_K$ in various limits. When $N \gg M \gg 1$ we find $\dim \mathcal{H}_K \sim 2^{2MN}$ to leading order. Thus in this limit, the dimension of the effective Hilbert space closely approximates the full Hilbert space of the original Grassmann system. For $M \gg N \gg 1$ we find instead $\dim \mathcal{H}_K \sim M^{N^2}$. Finally, for $M = \alpha N$ where α is fixed in the large N limit, we have:

$$\log \dim \mathcal{H}_K = f(\alpha) N^2 + \dots \quad (9.62)$$

with:

$$f(\alpha) = \frac{1}{2} (\alpha^2 \log(\alpha) - 2(\alpha + 1)^2 \log(\alpha + 1) + (\alpha + 2)^2 \log(\alpha + 2) - 2 \log 4) . \quad (9.63)$$

Notice that in the limit $\alpha \rightarrow 0$, $f(\alpha) \sim 2\alpha \log 2$ for which $\log \dim \mathcal{H}_K \sim 2NM \log 2$. Similarly, in the $\alpha \rightarrow \infty$ limit, $f(\alpha) \sim \log \alpha$ for which $\log \dim \mathcal{H}_K \sim N^2 \log M$. As shown in the appendix, (9.61) is precisely the number of states we would obtain in the Grassmann matrix model, had we gauged the $U(M)$ global symmetry. This is to be expected. The full space of $U(M)$ invariant states can be built by acting with a function of the $U(M)$ invariant operator $\hat{\mathbf{S}}_{ij}^+$ on the state $|v\rangle$ (which is itself defined to be $U(M)$ invariant by a suitable choice of the normal ordering constant in the $U(M)$ generators).

Hamiltonian and path integral

In the vector case, the Hamiltonian \hat{H} (9.4) we studied was constant along the Bloch two-sphere given that all the Bloch coherent states had the same total angular momentum. In this regard our matrix model differs from the vector case. Given our Hamiltonian operator (9.32), the Hamiltonian $H[Z, Z^\dagger] \equiv \langle Z | \hat{H} | Z^\dagger \rangle$ governing time evolution on the emergent classical phase space is found to be:

$$H[Z, Z^\dagger] = -NM^2 + M^2 \text{tr} (S^0)^2 , \quad (9.64)$$

to leading order in M . We have defined:

$$S^0 \equiv \left[(\mathbb{I} + ZZ^\dagger)^{-1} - (\mathbb{I} + Z^\dagger Z)^{-1} \right] . \quad (9.65)$$

Notice that $H[Z, Z^\dagger]$ is invariant under $Z \rightarrow UZU^\dagger$ where $U \in SU(N)$. Moreover, the Hamiltonian $H[Z, Z^\dagger]$ is minimized when Z and Z^\dagger commute, where it takes the value $E_{min} = -NM^2$. Consequently, the state $|v\rangle$ is one of these minimal energy states. This agrees with our analysis of the effective potential in section 9.3.2, where the minimum was also found to be $-NM^2$ in the large M limit. When Z and Z^\dagger commute they can be mutually diagonalized and the Kähler metric becomes N copies of \mathbb{CP}^1 , i.e. one Bloch sphere for each eigenvalue. Furthermore, as was found in the analysis of section 9.3.2, the commutator of Z and Z^\dagger costs energy. Nevertheless, since the Z can be continuously deformed, there is a rich low energy sector continuously connected to the ground states given by almost commuting complex matrices.

Given the kinetic term and the Hamiltonian on phase space, following Berezin [204], we can write down the real time path integral for transition amplitudes between coherent states $|Z_i^\dagger\rangle$ and $\langle Z_f|$. It reads:

$$\begin{aligned} \mathcal{A}_{fi} = \int \mathcal{D}\mu[Z, Z^\dagger] \exp \left(\frac{M}{2} \text{tr} \int_{-T}^T dt [\dot{Z}(\mathbb{I} + Z^\dagger Z)^{-1} Z^\dagger - h.c.] \right) \\ \cdot \exp \left(-i \int_{-T}^T dt H[Z, Z^\dagger] \right), \end{aligned} \quad (9.66)$$

with boundary conditions $Z^\dagger[-T] = Z_i^\dagger$ and $Z[T] = Z_f$. The measure factor is given by:

$$\mathcal{D}\mu[Z, Z^\dagger] \equiv \frac{1}{\mathcal{N}} \frac{\mathcal{D}Z \mathcal{D}Z^\dagger}{\det(\mathbb{I} + ZZ^\dagger)^{2N}}. \quad (9.67)$$

The normalization constant \mathcal{N} ensures that $\text{Tr} \mathbb{I} = \int d\mu[Z, Z^\dagger] = \dim \mathcal{H}_K$. It can be computed by use of the Selberg integral $S_N(1, M+1, 1)$ [230].

Consider finally the following rescaling $Z = M^{-1/2} \tilde{Z}$, with \tilde{Z} fixed in the large M limit, and in addition $M \gg N$. To leading order in the large M expansion, the path integral becomes:

$$\mathcal{A}_{fi} = \int \mathcal{D}\tilde{Z} \mathcal{D}\tilde{Z}^\dagger \exp \left[\frac{1}{2} \text{tr} \int_{-T}^T dt \left(\dot{\tilde{Z}} \tilde{Z}^\dagger - h.c. \right) - i \text{tr} \int_{-T}^T dt [\tilde{Z}, \tilde{Z}^\dagger]^2 \right]. \quad (9.68)$$

This limit is a small fluctuation limit in which the geometry of the curved phase space becomes flat and the Hamiltonian boils down to the trace of the square of the commutator. Naturally, in the $N = 1$ case, no such commutator arises, and the rescaling limit simply describes motion in a small flat patch of the full \mathbb{CP}^1 .

Thus, we generalize several of the features observed in the vector model to the matrix model. As before, there is an emergent classical phase space endowed with a Kähler metric, a low velocity expansion of a bosonic Hermitian matrix model in a suitable large M regime and a large number of low energy states. Given the appearance of a bosonic matrix model, we can wonder about a holographic interpretation at large N . We end with some speculative remarks on this question.

9.4 Outlook

We have discussed systems with a finite dimensional Hilbert space, whose constituents are a large number of spin-1/2 fermions. For certain collections of states, we have seen how the systems we have considered exhibit an emergent classical phase space parameterized by complex coordinates. The phase space is endowed with a Kähler metric which in the simplest case is nothing more than the round two-sphere. More generally, it is a complex matrix generalization thereof. In the vector case, the size of the Bloch sphere (9.29) scales as the logarithm of the dimension of the Hilbert space. The specific Hamiltonian we considered, commutes with the total angular momentum operator. Consequently, transition amplitudes between different Bloch coherent states lie on a Bloch sphere of fixed size. One manifestation of this is that the parameter r acquires no time derivatives in the effective action. More generally, one might imagine Hamiltonians with matrix elements connecting Hilbert spaces with different total angular momenta. In such a case, one might consider an additional direction given by the size of the two-sphere, such that in a suitable large M limit, the low energy degrees of freedom are parameterized by coordinates in a three-dimensional ball. So long as the dimension of the Hilbert space remains finite, there is still a cap on the maximal size of the two-sphere. A natural matrix generalization of the parameter r is given by the trace of the Hermitian matrix $\sqrt{\Sigma \cdot \Sigma}$. Unlike the vector case, transitions between different values of $\text{tr} \sqrt{\Sigma \cdot \Sigma}$ are possible within the space of Berezin coherent states. In other words, the Kähler metric of the emergent classical phase space does not constrain $\Sigma \cdot \Sigma$ (which is now a function of Z and Z^\dagger) to take a specific value.

Holographically, large N matrix models might be associated with a gravitational theory. For the quantum mechanical model [231] dual to the ten-dimensional geometry near a collection of N D0-branes, one has nine $N \times N$ Hermitian bosonic matrices \mathbf{X}_{ij}^I and their fermionic superpartners. The index I is an $SO(9)$ index, corresponding to the rotational symmetry of the eight-sphere in the near horizon of a stack of N D0-branes in type IIA string theory. The indices i and j run from 1 to N . The Hilbert space is infinite dimensional and

there are states with indefinitely high energy. In these models, the emergent radial direction has been argued to be captured by the energy scale. At high energies, the quantum mechanics is weakly coupled. One manifestation of this, from the bulk viewpoint, is that the size (in the string frame) of the eight-sphere shrinks indefinitely at large radial distances, eventually leading to a stringy geometry.

Consider now a system where the spectrum is capped, as occurs in the deep infrared of a CFT living on a spatial sphere (due to the curvature coupling of the fields). In such a situation we expect the emergent sphere to cap off. This is indeed what happens in global anti-de Sitter space where the sphere at fixed r and t smoothly caps off in the deep interior.⁸ Consider now the geometry of the static patch of four-dimensional de Sitter space:

$$ds^2 = -dt^2(1 - r^2) + \frac{dr^2}{(1 - r^2)} + r^2 d\Omega_2^2. \quad (9.69)$$

Notice that the size of the two-sphere resides on a finite interval. It smoothly caps off at $r = 0$ and is largest at $r = 1$ where the cosmological horizon resides. If, somehow, r was an emergent holographic direction related to the energy scale [232], then it would seem we have to cap the spectrum both in the infrared as well as the ultraviolet. This would indicate a holographic quantum mechanical dual with a finite number of states [218, 219, 220, 221, 222, 223], so long as the spectrum is discrete. If moreover we require the holographic model to have a matrix-quantum mechanical sector described by ordinary bosonic matrices, perhaps the systems we have considered above are natural candidates. We postpone the examination of this proposal and the relation to other approaches of de Sitter holography (for an overview see [233]) to future work.

⁸ Recall the metric of global AdS_{d+2} is given by $ds^2 = -dt^2(1+r^2) + dr^2(1+r^2)^{-1} + r^2 d\Omega_d^2$. As $r \rightarrow 0$ the d -sphere caps off smoothly.

Chapter 10

Holstein-Primakoff operators

We started in Chapter 8 by defining the Hilbert space (8.1), which we subsequently analyzed not in position space, but in the basis (8.4). In this chapter, we will relate back to the position space basis and attempt to identify the Hilbert space operators which, in the regime where the perturbative approximation is valid, reduce to the field operators of the higher spin fields in Vasiliev theory.

This introduction explains the different steps we have in mind to achieve this identification. We will see below that these steps cannot be executed exactly as stated here: the perturbative quantization of Vasiliev theory is not the exact UV-complete theory and breaks down beyond its regime of validity. We will nevertheless explain the program as if it could be realized exactly, and then analyze its breakdown later. Furthermore, the steps will be outlined in the opposite order of its execution: we will start by stating our goal, and then identify the steps necessary to get there.

- We want to find, in our Hilbert space, the Hermitian operators $\hat{\Phi}_{\mu_1 \dots \mu_s}^{(s)}(\eta, \vec{x})$ of the bulk higher-spin fields of Vasiliev theory.
- These operators can be characterized by their transformation properties under the isometries of de Sitter. The formalism of Chapter 7 is applicable if the Hilbert space contains creation and annihilation operators $a_{\mu_1 \dots \mu_s}^{(s)\dagger}(\vec{x})$ and $a_{\mu_1 \dots \mu_s}^{(s)}(\vec{x})$ for which the two-point functions

$$\langle 0 | a_{\mu_1 \dots \mu_s}^{(s)}(\vec{x}) a_{\mu_1 \dots \mu_s}^{(s)\dagger}(y) | 0 \rangle , \quad (10.1)$$

are those of a conformal spin- s current with weight $\Delta_s = 1 + s$ in a three-dimensional Euclidean CFT. The Vasiliev field operators could then be written as

$$\hat{\Phi}_{\mu_1 \dots \mu_s}^{(s)}(\eta, \vec{x}) = \int d^3 y [G(\eta, \vec{x}; \vec{y}) a_{\mu_1 \dots \mu_s}^{(s)}(\vec{x}) + G^\dagger(\eta, \vec{x}; \vec{y}) a_{\mu_1 \dots \mu_s}^{(s)\dagger}(\vec{x})] , \quad (10.2)$$

for appropriate functions G and G^\dagger . The problem identified in Chapter 7, would be avoided here: two-point functions are calculated as in perturbative field theory on de Sitter space, using (10.1) and the commutation relations of creation and annihilation operators. In particular, they are not automatically symmetric in $\eta \leftrightarrow \eta'$.

- The appropriate higher-spin creation and annihilation operators can be obtained from bilocal operators A_{xy} and A_{xy}^\dagger which satisfy

$$A_{xy} |0\rangle = 0 , \quad [A_{xy}, A_{zw}^\dagger] \propto \frac{1}{|x-w||z-y|} , \quad (10.3)$$

using the polynomials $D^{(s)}$ in (6.8),

$$a_{\mu_1 \dots \mu_s}^{(s)} z^{\mu_1} \dots z^{\mu_s} = D^{(s)}(z \cdot \partial_v, z \cdot \partial_w) A_{vw} \Big|_{v=x=w} , \quad (10.4)$$

and similar for the creation operators.

- If there were Hilbert space operators in the p, q basis which satisfy the Heisenberg algebra

$$[A_{pq}, A_{rs}^\dagger] = \delta_{ps} \delta_{qr} , \quad (10.5)$$

the operators appropriate A_{xy} and A_{xy}^\dagger could be obtained using the transformation (8.4).

However, we run into the problem that the Heisenberg algebra has no non-trivial representations on a finite dimensional Hilbert space. This can be seen by taking the trace of (10.5) and using that the trace of a commutator always vanishes in a finite dimensional Hilbert space. Therefore, canonically quantized Vasiliev theory can only be an approximation to this model with a limited regime of validity.

The goal of this chapter is to establish to what extent the commutation relations (10.5) can be approximated in our Hilbert space and how this approximation breaks down. We will start with the problem for $K = 1$ and consider the *Holstein-Primakoff transformation* [234], which relates spin raising and lowering operators to operators that approximately satisfy the Heisenberg algebra (see [235] for a review). We generalize the solution to $K > 1$ in §10.2. Finally, we say more about the breakdown of the Heisenberg algebra in §10.3.

10.1 K = 1

To obtain operators which approximately satisfy the Heisenberg algebra in the $K = 1$ Hilbert space, we will make use of the associated classical phase space, the Bloch sphere. The phase space analogs of Heisenberg operators are coordinates for which the Poisson bracket equals a constant. In other words, coordinates in which the symplectic form is of the form $dx_i \wedge dp^i$. Locally, the existence of such coordinates is guaranteed by Darboux's theorem of symplectic forms. We will therefore refer to them as Darboux coordinates. This can be expressed as $i du \wedge d\bar{u}$ in complex coordinates $x = (u + \bar{u})/\sqrt{2}$ and $p = (u - \bar{u})/\sqrt{2}i$. In the example of the Bloch sphere, these coordinates are readily obtained,

$$u \equiv \frac{z}{\sqrt{1 + \bar{z}z}} , \quad \bar{u} \equiv \frac{\bar{z}}{\sqrt{1 + \bar{z}z}} , \quad \omega^{(2)} = i du \wedge d\bar{u} = iN \frac{dz \wedge d\bar{z}}{(1 + \bar{z}z)^2} . \quad (10.6)$$

The inverse transformation is given by $z = u/\sqrt{1 - \bar{u}u}$. While z ranges over the full complex plane, the coordinate u is restricted to the region $|u|^2 < 1$. In terms of these coordinates then, the decomposition of the identity operator on the Hilbert space is

$$\hat{1} = \frac{N+1}{2\pi} \int_{|u|^2 < 1} du d\bar{u} |\bar{z}(u, \bar{u})\rangle \langle z(u, \bar{u})| , \quad (10.7)$$

where the states are the Bloch coherent states with z replaced by the appropriate function of the Darboux coordinates.

The functions u and \bar{u} in (10.6) can be interpreted as the symbols of some operators in the Hilbert space. For these operators, the symbol of their commutator is a constant up to leading order in the $1/N$ expansion. In other words, these operators approximately satisfy the Heisenberg algebra. At the level of symbols, one can see from (8.16) that

$$J^+ \left(\frac{1}{2} - \frac{J^3}{N} \right)^{-1/2} = u , \quad \left(\frac{1}{2} - \frac{J^3}{N} \right)^{-1/2} J^- = \bar{u} . \quad (10.8)$$

A reasonable ansatz would be that the operators \hat{u} and $\hat{\bar{u}}$ are given by these expressions in terms of the operators \hat{J}^\pm and \hat{J}^3 . The fractional powers of operators can formally be defined as their Taylor expansion around $J^3 = 0$. This is most conveniently checked in their representation as differential operators on holomorphic wave functions (8.14), which is intrinsically $U(N)$ invariant,

$$\mathcal{D}^{(u)} = Nz \left(1 - \frac{z}{N} \partial_z \right)^{1/2} , \quad \mathcal{D}^{(\bar{u})} = \left(1 - \frac{z}{N} \partial_z \right)^{-1/2} \partial_z . \quad (10.9)$$

Notice the positive power in $\mathcal{D}^{(u)}$ resulting from the form of $\mathcal{D}^{(J^+)}$ in (8.14). To check the claim, it suffices to calculate the commutator of these derivative operators. The term $\mathcal{D}^{(u)}\mathcal{D}^{(\bar{u})}$ is easily calculated to give $Nz\partial_z$. The other way around requires a little more work, specifically the observation that $z\partial_z$ commutes with powers of itself and the unit operator, and hence with $(1 - z\partial_z/N)^{\pm 1/2}$

$$\begin{aligned}\mathcal{D}^{(\bar{u})}\mathcal{D}^{(u)} &= N \left(1 - \frac{z}{N}\partial_z\right)^{-1/2} \partial_z z \left(1 - \frac{z}{N}\partial_z\right)^{1/2} \\ &= N \left(1 - \frac{z}{N}\partial_z\right)^{-1/2} (1 + z\partial_z) \left(1 - \frac{z}{N}\partial_z\right)^{1/2} \\ &= N(1 + z\partial_z) .\end{aligned}\tag{10.10}$$

The commutator of these differential operators equals N , which is constant. Since this representation is equivalent to the fermionic representation, at least insofar we restrict to the physical, $U(N)$ -invariant Hilbert space, we can conclude that

$$\hat{u} \equiv \hat{\mathcal{J}}^+ \left(\frac{1}{2} - \frac{\hat{\mathcal{J}}^3}{N} \right)^{-\frac{1}{2}} , \quad \hat{\bar{u}} \equiv \left(\frac{1}{2} - \frac{\hat{\mathcal{J}}^3}{N} \right)^{\frac{1}{2}} \hat{\mathcal{J}}^- , \tag{10.11}$$

act as “Darboux operators” on the $U(N)$ -invariant Hilbert space. This is the Holstein-Primakoff transformation [234], which relates spin raising and lowering operators to operators that satisfy the Heisenberg algebra (see [235] for a review).

An important note is in order. We have mentioned before that the Heisenberg algebra does not allow for finite dimensional representations. Yet we seem to have found a representation for each integer number of dimensions. Again the resolution is most easily stated in the language of holomorphic wave functions. Remember that the Hilbert space consists of polynomials in z up to order N . The maximally excited state z^N is annihilated by the differential operator $\mathcal{D}^{(u)}$. Therefore, the derivation (10.10) is flawed on this state and the result fails to hold. Instead, $[\mathcal{D}^{(\bar{u})}, \mathcal{D}^{(u)}]z^N = -N^2 z^N$.

This is an essential consequence of the underlying fermionic nature of the Hilbert space. It is of no consequence for states that have no component in the direction of the maximally excited state, but it can have very important consequences more generically, for example when tracing over all states in the Hilbert space.

10.2 The Berezin phase space

Each of the steps in the previous section can be generalized to the Hilbert space generated by (8.1) with some care. First, it is possible to define Darboux coordinates on the classical phase space $U(2K)/U(K) \times U(K)$,

$$U_{pq} = Z_{pr}(\mathbb{1} + Z^\dagger Z)^{-1/2}_{rq}, \quad U_{pq}^\dagger = (\mathbb{1} + Z^\dagger Z)^{-1/2}_{pr} Z_{rq}^\dagger. \quad (10.12)$$

The proof is more cumbersome than in the case $K = 1$. It can be worked out using the coordinates X^\dagger and Y^\dagger . We will omit it here, but we will outline the corresponding proof for the operators below.

At the level of operators, a similar Holstein-Primakoff (HP) transformation can be shown to exist. We will call $\hat{\mathcal{A}}^\dagger$ and $\hat{\mathcal{A}}$ the operators corresponding to the symbol U and U^\dagger , respectively. The reason for this (at first sight somewhat strange) choice to conform to the usual convention for Heisenberg creation and annihilation operators. Therefore we write

$$\hat{\mathcal{A}}_{pq}^\dagger \equiv \hat{\mathcal{J}}_{pr}^+ \left(1 - \frac{\hat{\mathcal{G}}}{N}\right)^{-1/2}_{rq}, \quad \hat{\mathcal{A}}_{pq} \equiv \left(1 - \frac{\hat{\mathcal{G}}}{N}\right)^{-1/2}_{pr} \hat{\mathcal{J}}_{rq}^-. \quad (10.13)$$

The fractional power of operators is meant to be expanded formally into the Taylor series of $(1 - x)^{-1/2}$. On holomorphic wave functions, one can see from (8.39) that they act as differential operators

$$\mathcal{D}_{pq}^{(\mathcal{A}^\dagger)} = N Z_{pr} (1 - G)_{rq}^{1/2}, \quad \mathcal{D}_{pq}^{(\mathcal{A})} = (1 - G)_{pr}^{-1/2} (\partial_Z)_{rq}, \quad (10.14)$$

where we use the shorthand $G_{pq} \equiv \mathcal{D}_{pq}^{(G)}/N$. Remember that we are using the convention $(\partial_Z)_{pq} = \partial/\partial Z_{qp}$. We will now proceed to show in this representation that $\mathcal{D}^{(\mathcal{A}^\dagger)}$ and $\mathcal{D}^{(\mathcal{A})}$ satisfy the Heisenberg algebra

$$[\mathcal{D}_{pq}^{(\mathcal{A})}, \mathcal{D}_{rs}^{(\mathcal{A}^\dagger)}] \approx N \delta_{ps} \delta_{rq}, \quad (10.15)$$

where \approx indicates that this equality fails on states which have support in the null space of the operator $(1 - \hat{\mathcal{G}})$ (see the caveat at the end of §10.1).

The proof is straightforward, but tedious. To start, observe that the right-hand side of (10.15) is contained in

$$\mathcal{D}_{pq}^{(\mathcal{A})} \mathcal{D}_{rs}^{(\mathcal{A}^\dagger)} = N (1 - G)_{pa}^{-1/2} [\delta_{rq} \delta_{ab} + Z_{rb} (\partial_Z)_{qa}] (1 - G)_{bs}^{1/2}. \quad (10.16)$$

It is therefore left to prove that the contribution from the second term in square brackets exactly cancels $\mathcal{D}_{rs}^{(\mathcal{A}^\dagger)} \mathcal{D}_{pq}^{(\mathcal{A})}$. Expanding the fractional powers in their

Taylor series – the coefficients are given by (analytically continued) binomial coefficients – this amounts to showing that

$$\begin{aligned} N \sum_{i,j \geq 0} \binom{-1/2}{i} \binom{1/2}{j} G_{pa}^i Z_{rb} (\partial_Z)_{qa} G_{bs}^j \\ \stackrel{?}{=} N \sum_{i,j \geq 0} \binom{-1/2}{i} \binom{1/2}{j} Z_{rb} G_{bs}^j G_{pa}^i (\partial_z)_{aq} . \end{aligned} \quad (10.17)$$

Starting from the second line, we first can commute $(ZG^j)_{rs}$ through G_{pa}^i . The commutator picked up can be calculated by induction, for example first in i and then in j , to give

$$\begin{aligned} [(ZG^i)_{rs}, G_{pa}] &= -\delta_{ps} (ZG^i)_{ra} , \\ [(ZG^i)_{rs}, G_{pa}^j] &= -\sum_{k=1}^j (G^{k-1})_{ps} (ZG^{i+j-k})_{ra} . \end{aligned} \quad (10.18)$$

In a second step, one can commute G_{bs} all the way to the right, through $(\partial_Z)_{aq}$. Again using induction, this is given by

$$[G_{bs}^i, (\partial_Z)_{aq}] = -\sum_{k=1}^i (G^{k-1} \partial_Z)_{bq} G_{as}^{i-k} . \quad (10.19)$$

These two operations map the second line of (10.17) onto the first line. We are thus left to show that the contribution from the commutators vanish,

$$\begin{aligned} 0 &\stackrel{?}{=} \sum_{i,j \geq 0} \binom{-1/2}{i} \binom{1/2}{j} \left(\sum_{k=1}^i G_{ps}^{k-1} (ZG^{i+j-k} \partial_z)_{rq} + G_{pa}^i (ZG^{k-1} \partial_Z)_{rq} G_{as}^{j-k} \right) \\ &= \sum_{i,j \geq 0} \binom{-1/2}{i} \binom{1/2}{j} \sum_{k=1}^{i+j} G_{ps}^{k-1} (ZG^{i+j-k} \partial_z)_{rq} . \end{aligned} \quad (10.20)$$

Where we have used the commutator $[G_{as}, (ZG^{k-1} \partial_Z)_{rq}] = 0$. This result vanishes for $i = j = 0$. Furthermore, it only depends on i and j through their sum, apart from the binomial coefficients. However, since the coefficients are those of inverse functions, one can see that the result also vanishes for $i + j > 0$,

$$1 = (1+x)^{1/2} (1+x)^{-1/2} = \sum_{i,j \geq 0} \binom{-1/2}{i} \binom{1/2}{j} x^{i+j} . \quad (10.21)$$

Since this must hold for all x , it must hold term by term in the expansion on the right-hand side, concluding the argument.

As in the $K = 1$ case, this derivation is invalid on states for which one of the operators $(1 - G)_{pq}$ vanishes. In particular, it cannot be used on sums over all states in the Hilbert space.

10.3 Hermitian HP operators

The Holstein-Primakoff operators (10.13) can be used to build states by acting with \mathcal{A}_{pq}^\dagger on the vacuum. When less than N creation operators are used, the Heisenberg algebra is satisfied exactly. Beyond that point it may break down, as a consequence of the finite dimensionality of the Hilbert space. Nevertheless, the HP operators seem to be well-suited candidate building blocks for the construction of bulk fields operators.

Let us therefore consider the following Hermitian combinations,

$$\hat{Q} \equiv \frac{\hat{A}^\dagger + \hat{A}}{\sqrt{2N}} , \quad \hat{P} \equiv \frac{\hat{A} - \hat{A}^\dagger}{\sqrt{2iN}} . \quad (10.22)$$

These operators are Hermitian as maxtrix operators in the Hilbert space, i.e. under the operation which maps the elementary fermionic operators $a \rightarrow a^\dagger$, $b \rightarrow b^\dagger$ and simultaneously changes the order of p, q matrix indices. The normalization with $1/N$ is for later convenience. Given the holographic correspondence $1/N \sim \Lambda G_N$, this normalization is sometimes referred to as the “supergravity normalization”. In theories with a Lagrangian description, it is the normalization for which the matter Lagrangian is proportional to the inverse of Newton’s constant. For states on which (10.15) applies, these operators satisfy

$$[\hat{Q}_{pq}, \hat{P}_{rs}] = \frac{i}{N} \delta_{ps} \delta_{rq} . \quad (10.23)$$

The commutator of two \hat{Q} operators or two \hat{P} operators vanishes, again only insofar (10.15) holds.

It is interesting to calculate the spectrum of these operators. There must be a finite number of eigenvalues, since the Hilbert space is finite-dimensional. However, if these operators are to be a good large- N approximation to the exact Heisenberg operators of perturbation theory, we should expect the set of eigenvalues to become dense in the classical limit. We will consider the case $K = 1$ first.

10.3.1 The simple case: $K = 1$

Everything can be worked out explicitly in this case. Consider the basis (8.8) for the Hilbert space. In our current model, we have $2j = N$ and the operators $\hat{J} \rightarrow \hat{\mathcal{J}}$ as in (8.6). The matrix elements of the HP operators (10.11) are

$$\begin{aligned}\langle m|\hat{u}|n\rangle &= \langle m|\hat{\mathcal{J}}^+ \left(1 - \frac{n}{N}\right)^{-1/2} |n\rangle = \sqrt{N(n+1)} \langle m|n+1\rangle , \\ \langle m|\hat{\bar{u}}|n\rangle &= \langle m|\left(1 - \frac{n}{N}\right)^{-1/2} \hat{\mathcal{J}}^- |n\rangle = \sqrt{N(m+1)} \langle m+1|n\rangle .\end{aligned}\quad (10.24)$$

Therefore, the \hat{Q} operator can be expressed in this basis as a matrix with only nonzero “superdiagonal” and “subdiagonal” elements, i.e. entries one unit away from the diagonal of the matrix. Its spectrum, and hence the spectrum of the operator \hat{Q} , is given by the solutions to the characteristic equation $\det(\lambda \mathbb{1} - \hat{Q}) = 0$. It will be convenient to rescale $\lambda = \tilde{\lambda}/\sqrt{2N}$ and solve the equation

$$\mathcal{P}_N(\tilde{\lambda}) = \det \begin{pmatrix} \tilde{\lambda} & -1 & 0 & \dots & 0 \\ -1 & \tilde{\lambda} & -\sqrt{2} & \ddots & 0 \\ 0 & -\sqrt{2} & \tilde{\lambda} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\sqrt{N} \\ 0 & 0 & 0 & -\sqrt{N} & \tilde{\lambda} \end{pmatrix} = 0 . \quad (10.25)$$

This determinant can be calculated recursively, for example by using the Laplace expansion on the rightmost row. Each of the minors can be expressed in terms of lower rank determinants of the same form. We get two terms,

$$\mathcal{P}_N(\tilde{\lambda}) = \tilde{\lambda} \mathcal{P}_{N-1}(\tilde{\lambda}) - N \mathcal{P}_{N-2}(\tilde{\lambda}) . \quad (10.26)$$

This is exactly the recurrence relation satisfied by Hermite polynomials $He_{N+1}(\tilde{\lambda})$. We can therefore conclude that the eigenvalues $\{\lambda\}$ of the \hat{Q} operator for $K = 1$ are given by the zeros of the Hermite polynomial¹ $H_{N+1}(\sqrt{N}\lambda)$.

The roots of the Hermite polynomials are well-understood in the regime of large N . In that limit, the Hermite polynomials are approximated by the following asymptotic expansion for $\lambda \ll \sqrt{2}$,

$$e^{-\frac{N}{4}\lambda^2} H_{N+1}(\sqrt{N}\lambda) \propto \cos \left(\sqrt{2N(N+1)} \lambda - \frac{(N+1)\pi}{2} \right) . \quad (10.27)$$

¹ From here on we will use the “physicists’ convention” Hermite polynomial $H_n(x)$ which is related to the “probabilists’ convention” by $H_n(x) = 2^{n/2} He_n(\sqrt{2}x)$.

Therefore, the eigenvalues are indeed uniformly spaced with separation $\Delta\lambda \approx \pi/\sqrt{2}N$.

We can borrow physical intuition from the quantum harmonic oscillator to understand this result. The wave functions of energy eigenstates are Hermite polynomials modulated by a Gaussian. It is well-known that these wave functions acquire an additional zero per energy level. Furthermore, in the [WKB](#) approximation, all roots of the wave function are in the classically allowed region. Indeed, the wave function behaves approximately as an exponential that decays when moving deeper into the classically forbidden region. It has no zeros there. In terms of the eigenvalues of \hat{Q} , the classically allowed region is $-\sqrt{2} < \lambda < \sqrt{2}$. Because of the symmetry between \hat{Q} and \hat{P} , the same conclusion holds for the eigenvalues of \hat{P} . In fact, one could have inferred this from the range of the symbols (10.6). The Hermitian combinations $q = (u + \bar{u})/\sqrt{2}$ and $p = (u - \bar{u})/\sqrt{2}i$ are analogous to \hat{Q} and \hat{P} in the sense that their star product is $1/N$, as in (10.23). Furthermore, the quantity $q^2 + p^2$ can be rewritten as $2\bar{u}u$, which is restricted between 0 and 2.

In this case with $K = 1$, it is also possible to calculate the corresponding eigenstates. One way to proceed is as follows. If the commutation relations (10.23) were exactly applicable on the full Hilbert space, the operator \hat{Q} would have a continuous spectrum, with eigenstates

$$\begin{aligned} |Q\rangle &\equiv e^{-\frac{N}{2}Q^2} e^{-\frac{1}{2N}\hat{u}^2 + \sqrt{2}Q\hat{u}} |0\rangle \\ &= e^{-\frac{N}{2}\text{Tr}(Q^2)} \sum_{n \geq 0} \frac{1}{n!} H_n(\sqrt{N}Q) \left(\frac{\hat{u}}{\sqrt{2N}} \right)^n |0\rangle . \end{aligned} \quad (10.28)$$

where we have used that the first line contains the generating function of Hermite polynomials. To verify that this state is indeed an eigenstate of the \hat{Q} operator (in this case given by (10.13) with $\mathcal{A}^\dagger \rightarrow \hat{u}$ and $\mathcal{A} \rightarrow \hat{\bar{u}}$) with eigenvalue Q , one uses the commutator (10.15). This is valid in the Bloch case on states with $0 \leq n \leq N$, because that involves the commutator acting on states up to $|N-1\rangle$. It goes wrong for $n = N+1$, since that state is not part of the Hilbert space (it is a state of zero norm) but if we were to use (10.15) it would incorrectly be mapped onto $|N\rangle$ by $\hat{\bar{u}}$. For states with $N+1 < n$ the commutator (10.15) can effectively be used as well, since those states are all outside of the Hilbert space and the $\hat{\bar{u}}$ operation maps null states on null states. The only problematic term in (10.28) is the one with $n = N+1$ and its coefficient vanishes whenever $H_{N+1}(\sqrt{N}Q) = 0$. Therefore, for these values of Q , they are exact eigenstates of the \hat{Q} operator.

10.3.2 The case of interest: $K > 1$

In the more general case of interest $K > 1$, most calculations are considerably more complicated. However, the argument with symbols instead of operators carries over rather straightforwardly. Consider the Hermitian matrices

$$\tilde{Q}_{pq} \equiv \frac{U_{pq} + U_{pq}^\dagger}{\sqrt{2}}, \quad \tilde{P}_{pq} \equiv \frac{U_{pq} - U_{pq}^\dagger}{\sqrt{2}i}. \quad (10.29)$$

These are the $K > 1$ analogs of p and q . The leading contribution to the star commutator $[\tilde{Q}, \tilde{P}]_\star$ is again proportional to i/N . Furthermore, the matrix $Q^2 + P^2 = UU^\dagger + U^\dagger U$ is bounded in the sense that its (real) eigenvalues must lie between 0 and 2. In the large N limit, we can therefore expect that the spectrum of the operators \hat{Q} and \hat{P} is bounded similarly by $\sqrt{2}$.

There is a calculation we can do to provide more evidence for this expectation, which involves the operators \hat{Q} and \hat{P} directly. Consider the operator product

$$\frac{2}{N^2} \hat{\mathcal{A}}_{pr}^\dagger \hat{\mathcal{A}}_{rq} = \hat{Q}_{pq}^2 + \hat{P}_{pq}^2 + i(\hat{Q}\hat{P} - \hat{P}\hat{Q})_{pq}. \quad (10.30)$$

There are two types of trace we can take. The one we have been talking about so far, $\text{Tr}(\hat{\mathcal{A}}_{pr}^\dagger \hat{\mathcal{A}}_{rq}) = \hat{\mathcal{A}}_{pr}^\dagger \hat{\mathcal{A}}_{rp}$ is the summation over the p, q indices. In other words, this is a sum of K^2 operators. The other kind of trace, which we will denote as $\text{Tr}_{\mathcal{H}}$, is a trace over the Hilbert space. It amounts to summing all the eigenvalues of the operator. Taking both traces of the previous equation, we get

$$\frac{2}{N^2} \text{Tr}_{\mathcal{H}}[\text{Tr}(\hat{\mathcal{A}}^\dagger \hat{\mathcal{A}})] = \text{Tr}_{\mathcal{H}}(\hat{Q}_{pq} \hat{Q}_{qp} + \hat{P}_{pq} \hat{P}_{qp} + i[\hat{Q}_{pq}, \hat{P}_{qp}]). \quad (10.31)$$

Careless implementation of the commutator (10.23) would yield the wrong result: it would replace the Hilbert space trace of the commutator by a term proportional to K^2 while instead it should vanish. Indeed, since the Hilbert space is finite-dimensional, traces of commutators are guaranteed to be zero. We can therefore set out to calculate the Hilbert space trace of $\text{Tr}(\hat{Q}^2 + \hat{P}^2)$. It can be expressed as an integral over the space of Berezin coherent states,

$$\begin{aligned} \text{Tr}_{\mathcal{H}}[\text{Tr}(\hat{Q}^2) + \text{Tr}(\hat{P}^2)] &= \frac{2}{N^2} \int [dZ] \text{Tr}[Z(\mathbb{1} + Z^\dagger Z)^{-1} Z^\dagger] \\ &= -\frac{\mathcal{N}}{N^2 K} \frac{\partial}{\partial \alpha} \int dZ dZ^\dagger \det(1 + \alpha Z^\dagger Z)^{-2K} \Big|_{\alpha=1}, \end{aligned} \quad (10.32)$$

where α is a real number and $\mathcal{N} = 1/\int [dZ]$. It is possible to rescale Z in such a way that all dependence on α is absorbed into a scale factor,

$$\text{Tr}_{\mathcal{H}}[\text{Tr}(\hat{Q}^2) + \text{Tr}(\hat{P}^2)] = -\frac{1}{N^2 K} \frac{\partial \alpha^{-K^2}}{\partial \alpha} = \frac{K}{N^2}. \quad (10.33)$$

This does not prove that for $K > 1$ there are no large eigenvalues of \hat{Q} and \hat{P} , but they must be sufficiently sparse to not significantly affect the variance we have computed.

It is possible to calculate the spectrum of these operators exactly for low values of K and N . We display some results in Figures 10.1 - 10.3.

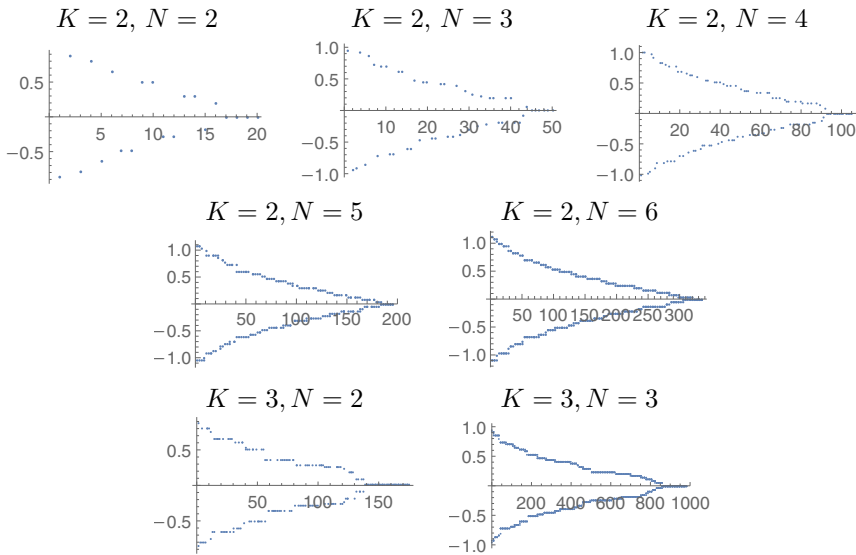


Figure 10.1: Spectrum of the \hat{Q}^{11} operator for the indicated values of (K, N) . The eigenvalues indexed in order of decreasing absolute value. This (arbitrary) index is displayed on the horizontal axis. The vertical axis gives the corresponding eigenvalue.

The eigenvalues of the Hermitian HP operators are indeed bounded in these examples. Because the Heisenberg algebra is not realized exactly on the full Hilbert space, the operators \hat{Q}_{pq} for different values of p and q do not commute with each other. This is illustrated in Figure 10.3, which displays the non-trivial eigenvalue spectrum of the commutator $[\text{Tr}(\hat{Q}), \text{Tr}(\hat{Q}^2)]$. This poses a challenge for the Hilbert space proposal. For example, it is not possible to define a wave function using a basis of eigenstates of the \hat{Q}_{pq} operators. Indeed, their failure to commute exactly means there is no such basis. Nevertheless, the Heisenberg commutator can still be used to calculate correlation functions with up to N insertions.

This leaves a very clear starting point and question for the program outlined at the start of this section, using the operators (10.15) as approximations to

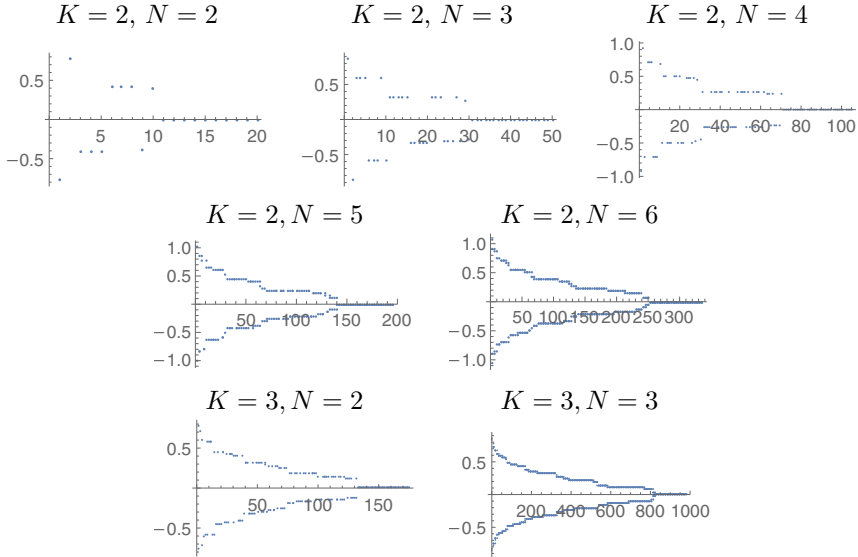


Figure 10.2: Spectrum of the operator $\frac{\hat{Q}^{12} + \hat{Q}^{21}}{2}$ for the indicated values of (K, N) . Their label is displayed on the horizontal axis, whereas the actual eigenvalue is on the vertical axis.

(10.5). Whether this program can be completed, given the breakdown of the Heisenberg algebra, will constitute the subject of future research.

We conclude with an incomplete list of specific questions that merit more research.

- Can higher-spin operators be defined consistently starting from the operators \mathcal{A}_{xy} and \mathcal{A}_{xy}^\dagger ?
- Can the Euclidean vacuum two-point functions of canonical quantization be reproduced?
- Is there a Hilbert space representation of the $|D\rangle$ and $|N\rangle$ vacuum states.
- Can the statements of dS/CFT be reproduced? Both the correlation functions as well as the statement in terms of generating functions. Does the bound on the spectrum of \hat{Q} and \hat{P} resolve the question of normalizability of the wave function?
- Is this proposal consistent with the physical requirement that local operators should commute at spacelike separation, or does the failure of

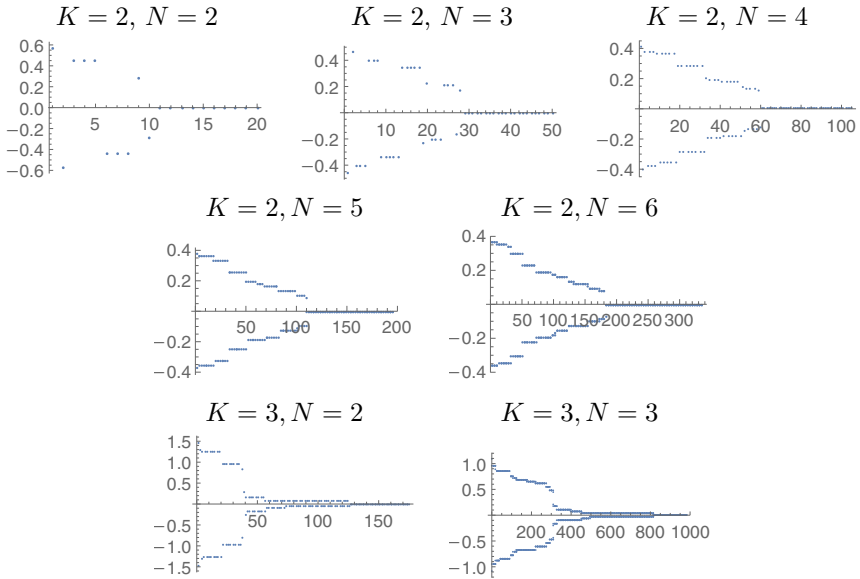


Figure 10.3: Spectrum of the operator $[\text{Tr}\hat{Q}, \text{Tr}\hat{Q}^2]$ for the indicated values of (K, N) . Their label is displayed on the horizontal axis, whereas the actual eigenvalue is on the vertical axis.

the \hat{Q} and \hat{P} operators to satisfy the exact Heisenberg algebra pose a problem?

Chapter 11

Conclusions

We have addressed only a small part of the countless interesting aspects of the physics in AdS and dS spacetimes. A number of concrete open questions and directions of further research have already been mentioned during the discussion of each of the topics we presented. We will use this section to give a summary and present a more general outlook on possible future research.

In Chapter 2 we reviewed established results about the physics in AdS and their holographic link to conformal field theory. The analysis of Chapter 3 was presented in this context. Using numerical simulations, we found black hole solutions surrounded by non-trivial matter field configurations in the M-theory compactification M^{11} . As discussed, this can be the starting point to embed the holographic description of vitrification into M-theory. The numerical tools we used can be applied to more general supergravity theories, whether embedded in M-theory or not. It would be useful to obtain a more general overview of non-BPS black hole solutions in AdS. This can describe field theory phenomena at strong coupling via the AdS/CFT correspondence, both in thermal equilibrium states as well as out of equilibrium.

In Chapter 4, we presented and discussed the proposal of dS/CFT, highlighting the differences with holography in AdS. Chapter 5 was dedicated to the analysis of the no-boundary wave function of the Universe in the basis of asymptotically finite variables. We identified the condition for the wave function in this basis to predict classical evolution, and found that it is more constraining than previous conditions. The analysis in this chapter can be generalized to different dimensions and more general matter couplings. It would be interesting to investigate in these more general settings which branches of the cosmological wave function are selected by this asymptotic classicality condition.

The first explicit models of dS/CFT are discussed in Chapter 6. The bulk side of these dualities are Vasiliev's higher spin theories. A natural question to ask is if there are models which instead describe the matter content of the Standard Model of particle physics. Since these fields are considerably more massive than the energy scale set by the cosmological constant, the Euclidean CFT dual to our Universe should contain operators of large complex dimension $\Delta = d/2 + i\delta$, with $\delta \in \mathbb{R}$. Such a theory is very different from CFTs used in the description of unitary Lorentzian theories, but the spectrum of operators is still constrained to a one real parameter family of conformal weights. It would be very interesting to derive constraints on the space of possible CFTs dual to de Sitter space, in analogy with the conformal bootstrap for reflection positive theories. This is a well-defined mathematical question which would impose direct restrictions onto the space of possible quantum gravitational theories that can be obtained through the dS/CFT correspondence.

The results of Chapter 7 specify the notion of bulk locality for scalar operators in dS/CFT. The procedure can be extended to operators of other spins. This already has a direct application in the program of Chapter 10. The construction we have outlined is only valid in perturbation theory around empty de Sitter space. An important open question is whether this construction can be generalized beyond this regime.

The model presented and analyzed in Chapter 8 - 10 was proposed to address the challenges for dS/CFT identified in Chapters 6 and 7. The pivotal element of the model is the explicit Hilbert space for the bulk. The gauge symmetry and vector representation of the AHS model were preserved in order to match the spectrum of Vasiliev theory in dS. We described in Chapter 8 how the model gives rise to an emergent compact classical phase space on which holomorphic wave functions can be defined. The Hilbert space provides a unique and positive definite inner product on the space of wave functions. In Chapter 9 we presented a model of matrix quantum mechanics which associates dynamical content with this theory. In Chapter 10 we addressed the important question of identification of bulk operators in this Hilbert space. Because of the finite dimensionality of the Hilbert space, it is not possible to embed the infinite Hilbert space of canonical quantization. This was expressed as the inability to represent the Heisenberg algebra exactly on the Hilbert space. We identified an approximate embedding and described when it breaks down. Our analysis has lead to a number of concrete questions and has provided an explicit setting for calculations to be performed. These will be the subject of further research.

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